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Complete 1-loop test of AdS/CFT

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Abstract: We analyze nested Bethe ansatz (NBA) and the corresponding finite size corrections. We find an integral equation which describes these corrections in a closed form. As an application we considered the conjectured Beisert-Staudacher (BS) equations with the Hernandez-Lopez dressing factor where the finite size corrections should reproduce generic one (worldsheet) loop computations around any classical superstring motion in the $AdS_5 \times S^5$ background with exponential precision in the large angular momentum of the string states. Indeed, we show that our integral equation can be interpreted as a sum over all physical fluctuations and thus prove the complete 1-loop consistency of the BS equations. In other words we demonstrate that any local conserved charge (including the AdS Energy) computed from the BS equations is indeed given at 1-loop by the sum of charges of fluctuations up to exponentially suppressed contributions in the large angular momentum os the string states. We also point out where precisely we loose the exponential terms in our ab initio analysis. Contrary to all previous studies of finite size corrections, which were limited to simple configurations inside rank 1 subsectors, our treatment is completely general.

Keywords: AdS-CFT Correspondence, Integrable Field Theories, Bethe Ansatz.

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1. Introduction

Bethe equations [1] describe the scattering of the fundamental degrees of freedom of integrable $1+1$ dimensional theories defined on some large circle of length $\mathcal{L}$. The existence of a large amount of conserved charges results in the factorizability property of the scattering matrix. Namely the full n particle S-matrix is completely fixed by the 2 particle scattering. Moreover in two dimensions this 2 to 2 scattering process conserves not only the total momentum but also the set of individual momenta. Then, for a large enough circle, the momenta of the several particles are quantized through the wave function periodicity condition

$$1 = e^{ip_k \mathcal{L}} \prod_{j \neq k}^{L} S(p_k, p_j) \tag{1.1}$$

meaning that the (trivial) phase acquired by a particle with momentum p_k while going around the circle equals the free propagation plus the scattering phases shifts (or time delay in coordinate space) due to the passage through each of the other particles. In general Bethe equations are only asymptotically exact as $\mathcal{L} \to \infty$ otherwise wrapping effects [2, 3] must be taken into account.

Equation (1.1) is, however, describing particles with no isotopic degrees of freedom, that is $S(p_k, p_j)$ is just a phase. In general, when we have some nontrivial symmetry group, this is not the case and, rather, we must solve the diagonalization problem

$$|\psi\rangle = e^{ip_k \mathcal{L}} \prod_{j \neq k}^{L} \hat{S}\left(p_k, p_j\right) |\psi\rangle$$

where $\hat{S}(p_k, p_j)$ is now a matrix and $|\psi\rangle$ is the multi-particle wave function (for integrable theories the number of particles is conserved). If the scattered particles transform under some symmetry group we will obtain not just one equation like (1.1) but rather a set of n equations entangling the scattering of particles with momenta p_k and p_j in space-time with the scattering of spin waves in the isotopic space.

In this paper we will mainly consider the particular limit of low energies when the wave length of the spin waves are large and particles exhibit collective behavior which, in some important cases, can be associated with the classical motion of collective fields. By studying carefully this limit one can get important information about the quantization of some classical field theories.

In terms of the Bethe ansatz equation this corresponds to a limit, first considered in the condensed matter literature by Sutherland [4] in the study of the ferromagnetic limit of the Heisenberg chain and rediscovered and generalized in the context of AdS/CFT [5], where the Bethe roots $u_j \sim \cot(p_j/2)$ scale with the number of such roots and with the total number of particles, $u_j \sim K_a \sim L$. In this limit the Bethe roots condense into disjoint cuts. Since there are several types of Bethe roots, one for each Bethe equation, the condensation of the Bethe roots for systems with n Bethe equations will generate some Riemann surface with $n+1$ sheets as in figure 4. This resulting curve is in one-to-one correspondence with the curves classifying classical solutions through the finite gap method [6 − 10]. In this way one finds the semi-classical spectrum of the theory.

The next natural step is to compute the first quantum corrections to the semi-classical spectrum, which from the Bethe ansatz point of view will correspond to the finite size (i.e. $1/L$) corrections. For the simplest Bethe equations of the form (1.1) these corrections, called anomalies, were known [11 − 17] but for nested Bethe ansatz equations the analysis is much more delicate due to the formation of bound states, called stacks [18], which are the basic constituents of the cuts made out of more than one type of Bethe roots like the ones in figure 3. In this paper we develop the necessary tools to deal with these richer systems with isotopic degrees of freedom.

Particularly important tools are the so called *dualities*. One of them, the *fermionic* duality, is well studied [18 − 24] and has a clear mathematical meaning. If the symmetry group under which the fundamental particles transform is a super group then there are several possible choices of NBA equations corresponding to the several possible choices of super Dynkin diagram which, for super-groups, is not unique. These equations are related by some dualities associated with the fermionic nodes of the corresponding super Dynkin diagram. In the scaling limit they correspond to the exchange of Riemann sheets. In this paper we also use an analogue of this duality, baptized *bosonic* duality, which exists even in the case of a purely bosonic symmetry. It is associated with the bosonic nodes of the Dynkin diagram.

Next we apply our method to the recently conjectured Beisert-Staudacher (BS) Bethe equations [25]. These equations contain a free parameter λ and should describe two systems at the same time: four dimensional $\mathcal{N}=4$ SYM and type IIB super-strings in $AdS_5 \times S^5$, two theories which are conjectured to be dual [26 − 28]. At weak coupling, $\lambda \ll 1$, we are in the perturbative regime of $\mathcal{N}=4$ SYM and the Bethe equations describe the spectrum of the planar dilatation operator which can be considered as a spin chain Hamiltonian [29, 30] with PSU$(2,2|4)$ symmetry. At strong coupling $\sqrt{\lambda} \sim L \gg 1$ the theory describes classical super-strings in the curved space-time $AdS_5 \times S^5$ [8, 18] and the $1/\sqrt{\lambda}$ corrections in the scaling limit correspond to the semi-classical quantization of such highly non-trivial quantum field theory.

As it was stressed in our previous papers [31, 32] there are two completely different ways to compute the 1-loop correction to the quasi-classically quantized energy spectrum. One, straightforward but technically more involved, is to take the NBA equations, compute its spectrum and then expand it in powers of $1/\sqrt{\lambda}$ i.e. find its finite size corrections. Another way, more indirect one, is to pick some classical solution satisfying the semi-classical quantization condition, and quantize around it, i.e. find the spectrum of all possible excitations of this solution. The one loop shift is then given by the zero energy oscillations and is equal to half of the graded sum of all excitation energies, like for a simple set of independent one dimensional harmonic oscillators.

Both calculations can be performed using the BS equations and it is a very nontrivial

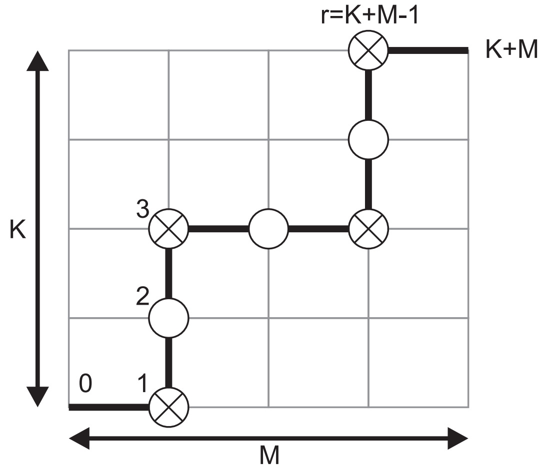

Figure 1: For $su(K|M)$ super algebras the Dynkin diagram is not unique. The several possible choices can be represented as the paths going from the up right corner (M, K) to the origin always approaching this point with each step. The turns are the fermionic nodes whereas the straight lines correspond to the usual bosonic nodes. Different paths will correspond to different sets of Bethe equations which are related by fermionic dualities which flip a *left-down* fermionic turn into *down-left* turn or vice-versa [24].

test of the proposed equations that these two calculations give the same result. In fact for the second calculation we do not even need the Bethe ansatz, since it is based completely on the semi-classical quantization which, as shown in [31], can be performed relying uniquely on the classical integrable structure of the theory — the algebraic curve [8, 9]. Moreover we expect the second approach to give the exact result whereas the first one is only valid as long as one can trust the asymptotic BAE, which suffers from the wrapping effects mentioned above. Indeed we found that the two results coincide not precisely, but only for large $L/\sqrt{\lambda}$ with exponential precision. This is obviously a manifestation of the wrapping effects considered in the AdS/CFT context in [33 − 37]. This exponential mismatch was first observed in [38].

Finally we should stress that we follow a constructive approach. That is we start from the classical integrable structure, the finite gap curves. The curves can be described by some integral equations. We find how to correct this equations *in such a way* that they will now describe not only the classical limit $\sqrt{\lambda} \to \infty$ but also the $1/\sqrt{\lambda}$ corrections. Then we show that the integral equations modified in this way coincide precisely with the scaling limit expansion of the BS equations [25] with the Hernandez-Lopez phase [39] (up to some exponentially suppressed wrapping effects, irrelevant for large angular momentum string states)! Our comparison, being done at the functional level, is completely general.

This paper is organized as follows. In section 2 we introduce some notations, the notion of stack and the bosonic duality. In section 3 we derive, in two independent ways, an integral equation describing the finite size corrections to the leading limit - using the dualities and using the transfer matrices. In section 6 we follow the constructive approach mentioned above to re-derive the same integral equation from the equations in the scaling limit. Section 5 contains some details about the bosonic duality such as some theorems and

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examples — the reader interested only in the main results of the paper can skip this section. In section 6 we apply the methods developed in the previous sections to the study of the BS equations, compute the finite size corrections and relate them with the quantum fluctuations of the theory. Appendix A is devoted to the study of the invariance of the transfer matrices of $su(K|M)$ supergroups under the bosonic duality and in appendix B we derive an integral equation describing the semi-classically corrected equations for $su(n)$ spin chains.

2. Nested Bethe ansatz and bosonic duality

In the first sections we stick mainly to the simple example of $su(1,2)$ spin chain, although our main motivation comes from its application to $AdS_5 \times S^5/\mathcal{N} = 4$ SYM correspondence where the symmetry group is PSU$(2,2|4)$. Indeed this simple toy model contains already all the nontrivial new features appearing due to the Nested nature of the Bethe ansatz. The generalization to other (super)groups is straightforward and in particular we shall focus on the Bethe ansatz describing the superstring in $AdS_5 \times S^5$ in section 6.

For integrable rank r spin chains each quantum state is parameterized by a set $\{u_{a,j}\}$ of Bethe roots where $a = 1, \ldots, r$ refers to the Dynkin node and $j = 1, \ldots, K_a$ where K_a is the excitation number of *magnons* of type a. The Bethe equations from which we find the position of these roots are then given by

$$e^{i\tau_a} \left(\frac{u_{a,j} + \frac{i}{2} V_a}{u_{a,j} - \frac{i}{2} V_a} \right)^L = - \prod_{b=1}^{r} \frac{Q_b \left(u_{a,j} + \frac{i}{2} M_{ab} \right)}{Q_b \left(u_{a,j} - \frac{i}{2} M_{ab} \right)} \tag{2.1}$$

where

$$Q_a(u) = \prod_{j=1}^{K_a} (u - u_{a,j})$$

are the Baxter polynomials, V_a are the Dynkin labels of the representation considered and M_{ab} the Cartan matrix. In fact, contrary to what happens for the usual Lie algebras, for super algebras the Dynkin diagram (and thus the Cartan matrix) is not a unique. Take for example the $su(K|M)$ super algebra. The different possible Dynkin diagrams can be identified [24] as the different paths starting from (M, K) and finishing at $(0,0)$ (always approaching this point with each step) in a rectangular lattice of size $M \times K$ as in figure 1. The turns in this path represent the fermionic nodes whereas the bosonic nodes are those which are crossed by a straight line — see figure 1 (the index a goes along the path as indicated). The Cartan matrix M_{ab} is then given by

$$M_{ab} = (p_a + p_{a+1})\, \delta_{ab} - p_{a+1} \delta_{a+1,b} - p_a \delta_{a,b+1}$$

where p_a is associated with the link between the node a and $a+1$ and is equal to $+1$ (-1) if this link is vertical (horizontal).

Here we are considering twisted (quasi-periodic) boundary conditions which, from an algebraic Bethe ansatz point of view corresponds to the diagonalization of a transfer matrix

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with the insertion, inside the trace, of an additional diagonal matrix [40] which can be parameterized by

$$g = \text{diag}\left(e^{i\phi_1}, \ldots, e^{i\phi_K}, e^{i\varphi_1}, \ldots, e^{i\varphi_M}\right) \in \text{SU}(K|M) \tag{2.2}$$

and the twists τ_a, appearing in (2.1) and associated to a Dynkin node located at (m,k) in the $M \times K$ network depicted in figure 1, are then given by [40]

$\tau_a = \phi_k - \phi_{k+1}$ *for a bosonic along a vertical segment of the path*
$\tau_a = \varphi_{m+1} - \varphi_m$ *for a bosonic along a horizontal segment of the path*
$\tau_a = \varphi_{m+1} - \phi_k + \pi$ *for a fermionic node in a* Γ *like turn that is with* $p_{a-1} = -p_a = 1$
$\tau_a = \phi_{k+1} - \varphi_m + \pi$ *for a fermionic node with* $p_{a-1} = -p_a = -1$

Notice that since $g \in \text{SU}(K|M)$ we have $\sum_k \phi_k - \sum_m \varphi_m = 0 \mod 2\pi$. We shall study these Bethe equations with generic twists and we will see that the usual case ($\tau_a = 0$) is in fact quite degenerate.

As mentioned above, we find already all the ingredients we will need for the study of the BS equations in the simple example of a $su(1,2)$ spin chain in the fundamental representation described by the following system of NBA equations[1]

$$e^{i\phi_1 - i\phi_2} = -\frac{Q_1(u_{1,j}+i)}{Q_1(u_{1,j}-i)}\frac{Q_2(u_{1,j}-i/2)}{Q_2(u_{1,j}+i/2)}\,, \qquad j = 1\ldots K_1 \tag{2.3}$$

$$e^{i\phi_2 - i\phi_3}\left(\frac{u_{2,j}-\frac{i}{2}}{u_{2,j}+\frac{i}{2}}\right)^L = -\frac{Q_2(u_{2,j}+i)}{Q_2(u_{2,j}-i)}\frac{Q_1(u_{2,j}-i/2)}{Q_1(u_{2,j}+i/2)}\,, \qquad j = 1\ldots K_2 \tag{2.4}$$

The eigenvalues of the local conserved charges are functions of the roots $u_{2,j}$ only and are given by

$$\mathcal{Q}_r = \sum_{j=1}^{K_a} \frac{i}{r-1}\left(\frac{1}{(u_{2,j}+i/2)^{r-1}} - \frac{1}{(u_{2,j}-i/2)^{r-1}}\right). \tag{2.5}$$

We will often denote these roots carrying charges by *middle node* roots.

First, consider only middle node excitations, $K_1 = 0 \neq K_2$ in the Sutherland scaling limit [4] where $u \sim K_2 \sim L \gg 1$. We shall always use $x_{a,j} = u_{a,j}/L$ to denote the rescaled Bethe roots in the scaling limit. Then, the Bethe equations in log form, to the leading order, can be cast as

$$2\pi n_j + \phi_2 - \phi_3 = \frac{1}{x_{2,j}} + \frac{2}{L}\sum_{k\neq j}\frac{1}{x_{2,j} - x_{2,k}} \tag{2.6}$$

where the integers n_j come from the choice of the branch of the logs.

We see that we can think of the Bethe roots as positions of 2d Coulomb charges on a plane with an external potential equal for every particle plus an external force $2\pi n_j$ specific of each Bethe root. Thus, if we group the K_2 mode numbers $\{n_j\}$ into N large groups of

[1] These equations are exactly the same as for the $su(3)$ spin chain except for the sign of the Dynkin labels which makes the system simpler because the Bethe roots are in general real.

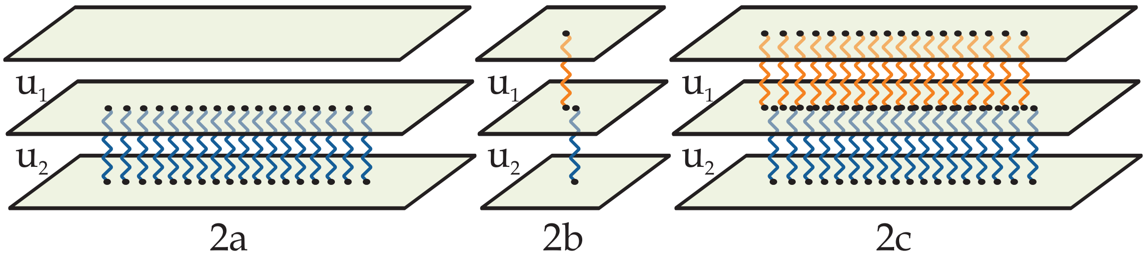

Figure 2: The *middle node* Bethe roots u_2 can condense into a line as depicted in figure 2a (The spins in this spin chain transform in a non-compact representation and thus the cuts are topically real. For the $su(2)$ Heisenberg magnet the solutions are distributed in the complex plane as some *umbrella* shaped curves [5].). Roots of different types can form bound states, called stacks [18], as shown in figure 2b. The stacks behave as fundamental excitations and can also form cuts of stacks as represented in figure 2c.

identical integers and consider the limit where both L and K_2 are very large with K_2/L fixed, the Bethe roots will be distributed along N (real) cuts $\mathcal{C}^A$, each parameterized by a specific mode number $\{n^A\}$ where $A = 1, \ldots, N$. Then the equations (2.6) can be written through the density of *middle node* roots x_2 as

$$2\pi n^A + \phi_2 - \phi_3 = \frac{1}{x} + 2\,\not{G}_2(x)\ , \ \ x \in \mathcal{C}^A \tag{2.7}$$

where we introduce the resolvents

$$G_a(x) = \int \frac{\rho_a(y)}{x-y} \ \ , \ \ \rho_a(y) = \frac{1}{L}\sum_{j=1}^{K_a} \delta(x - x_{a,j}) \tag{2.8}$$

and where the slash of some function means the average of the function above and below the cut, $\not{G}(x) = \frac{1}{2}\left(G(x+i\epsilon) + G(x-i\epsilon)\right)$. Let us also introduce some notation useful for what will follow. Defining the quasi-momenta as

$$\begin{aligned} p_1 &= -\frac{1}{2x} + G_1 - \phi_1\,, \\ p_2 &= -\frac{1}{2x} - G_1 + G_2 - \phi_2\,, \\ p_3 &= -\frac{3}{2x} - G_2 - \phi_3\,, \end{aligned} \tag{2.9}$$

we can add the indices 23 to the mode number n^A and to the cut $\mathcal{C}^A$ in (2.7) and recast this equation as

$$2\pi n_{23}^A = \not{p}_2 - \not{p}_3\ , \ \ x \in \mathcal{C}_{23}^A\,. \tag{2.10}$$

Next let us consider a state with only two roots $u_{2,1} \equiv u$ and $u_{1,1} \equiv v$ with different flavors, that is $K_1 = K_2 = 1$. Bethe equations then yield

$$u = \frac{1}{2}\cot\frac{\phi_1 - \phi_3 + 2\pi n}{2L}\ \ , \ \ v = u + \frac{1}{2}\cot\frac{\phi_1 - \phi_2}{2} \tag{2.11}$$

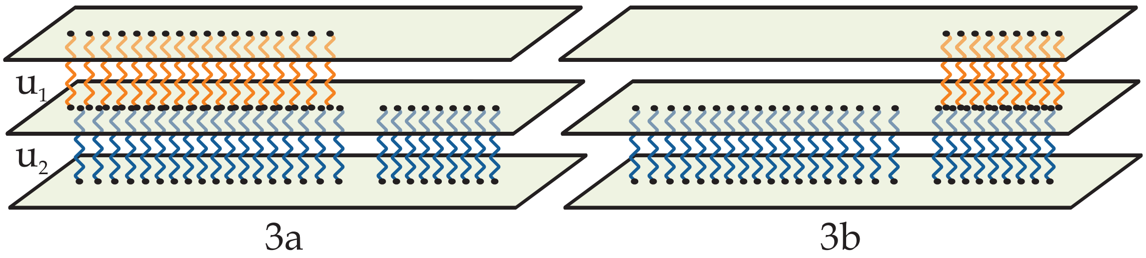

Figure 3: In the scaling limit, to the leading order, the bosonic duality reads $Q_2 \simeq Q_1\tilde{Q}_1$ with $Q_a = \prod_{k=1}^{K_a}(u - u_a)$. Thus, if we start with the configuration in figure 3a where the K_1 roots u_1 form a cut of stacks together with K_1 out of the K_2 middle node roots u_2 and apply the bosonic duality to this configuration, the $K_2 - K_1$ new roots $\tilde{u}_1$ must be close to the roots u_2 which were previously *single* while the cut of stacks in the left of figure 3a will become, after the duality, a cut of simple roots — see figure 3b.

which tells us that if $n \sim 1$ we are in the scaling limit where $v \sim u \sim L$ and $v = u + \mathcal{O}(1)$ — the two Bethe roots form a bound state, called stack [18], and can be thought of as a fundamental excitation — see figure 2*b*. On the other hand we notice that, strictly speaking, for the usual untwisted Bethe ansatz with $\phi_a = 0$ the stack no longer exists.

Since the stack in figure 2*b* seems to behave as a fundamental excitation one might wonder whether there exists a cut with $K_1 = K_2$ roots of type u_1 and u_2, like in figure 2*c*, *dual* to the configuration plotted in figure 2*a*. To answer affirmatively to this question let us introduce a novel kind of duality in Bethe ansatz which we shall call *bosonic duality*.

Indeed, as we explain in detail in section 5, given a configuration of K_1 roots of type u_1 and K_2 roots of type u_2, we can write

$$2i \sin(\tau/2)\, Q_2(u) = e^{i\tau/2} Q_1(u - i/2)\tilde{Q}_1(u + i/2) - e^{-\tau/2} Q_1(u + i/2)\tilde{Q}_1(u - i/2)\,, \quad (2.12)$$

where

$$\tilde{Q}_1(u) = \prod_{j=1}^{\tilde{K}_1} (u - \tilde{u}_{1,j}) \;\;, \;\; \tilde{K}_1 = K_2 - K_1\,,$$

and $\tau = \phi_1 - \phi_2$. Moreover this decomposition is unique and thus defines unambiguously the position of the new set of roots $\tilde{u}_1$. Then, as we explain in section 5, the new set of roots $\{\tilde{u}_1, u_2\}$ is a solution of the same set of Bethe equations (2.1) with

$$\phi_1 \leftrightarrow \phi_2\,.$$

Let us then apply this duality to a configuration like the one in figure 2*a* where the roots $u_2 \sim L$ are in the scaling limit and where there are no roots of type u_1, $K_1 = 0$. To the leading order, we see that the $\tilde{u}_1$ in (2.12) will scale like L so that the $\pm i/2$ inside the Baxter polynomials can be dropped and we find $Q_2 \simeq \tilde{Q}_1$, that is

$$\tilde{u}_{1,j} = u_{2,j} + \mathcal{O}(1)$$

and therefore we will indeed obtain a configuration like the one depicted in figure 2*c*. Moreover the local charges (2.5) of this dual cut are exactly the same as those of the

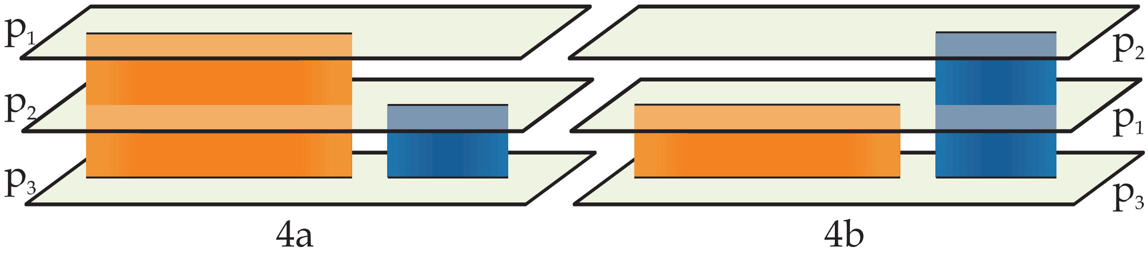

Figure 4: In the scaling limit the configurations in figure 3 condense into some disjoint segments, cuts, and we obtain a Riemann surface whose sheets are the quasi-momenta. In this continuous limit the duality corresponds to the exchange of the Riemann sheets.

original cut 2a since they are carried by the *middle node* roots u_2 which are untouched during the duality transformation.

Finally, if we apply the duality transformation to some configuration like that in figure 3a in the scaling limit we find, by the same reasons as above, that $Q_2(u) \simeq Q_1(u)\tilde{Q}_1(u)$. This means that the dual roots $\tilde{u}_1$ will be close to the roots u_2 which are not yet part of a stack — the ones making the cut in the right in figure 3a. Thus, after the duality, we will obtain a configuration like the one in figure 3b.

We conclude that, in the scaling limit with a large number of roots, the distributions of Bethe roots condense into cuts in such a way that the quasi-momenta $p_{\rm i}$ introduced above become the three sheets of a Riemann surface, see figure 4a, obeying

$$2\pi n_{\rm ij}^A = \not{p}_{\rm i} - \not{p}_{\rm j} \ , \ x \in \mathcal{C}_{\rm ij}^A \, . \tag{2.13}$$

when x belongs to a cut joining sheets i and j with mode number $n_{\rm ij}^A$. The duality transformation amount to a reshuffling of sheets 1 and 2 of this Riemann surface[2] so that a surface like the one plotted in figure 4a transforms into the one indicated in figure 4b.

3. Anomalies — finite size correction to nested Bethe ansatz equations

In this section we will study the leading $1/L$ corrections to the scaling equations (2.13). Moreover since the charges of the solutions are expressed through *middle node roots* u_2 and since these roots are duality invariant it is useful to write the Bethe equations in terms of these roots only. Let us then consider a given configuration of roots condensed into some simple cuts $\mathcal{C}_{23}$ and some cuts of stacks $\mathcal{C}_{13}$. Then, to leading order, at cuts $\mathcal{C}_{23}$ we have

$$\frac{1}{x} + 2 \fint_{\mathcal{C}_{23}} \frac{\rho_2(y)dy}{x-y} + \int_{\mathcal{C}_{13}} \frac{\rho_2(y)dy}{x-y} = 2\pi n_{23}^A + \phi_2 - \phi_3 \ , \ x \in \mathcal{C}_{23} \tag{3.1}$$

because in a cut $\mathcal{C}_{13}$ we have $\rho_1 \simeq \rho_2 + \mathcal{O}(1/L)$. To study finite size corrections to this equation two contributions must be considered. On the one hand when expanding the *self*

[2]As we shall see in the next section this interpretation can be made exact, and not only valid in the scaling limit.

interaction we get [11–16]

$$\sum_{j\neq k} i\log\frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i}=2\,\fint_{\mathcal{C}_{23}}\frac{\rho_2(y)dy}{x-y}+2\int_{\mathcal{C}_{13}}\frac{\rho_2(y)dy}{x-y}+\frac{1}{L}\pi\rho_2'\cot\pi\rho_2$$

where the $1/L$ correction comes from the contribution to the sum from the roots separated by $\mathcal{O}(1)$. On the other hand the auxiliary roots appear as[3]

$$\sum_{j} i\log\frac{u_{2,k}-u_{1,j}+i/2}{u_{2,k}-u_{1,j}-i/2}=-\int_{\mathcal{C}_{13}}\frac{\rho_1(y)}{x-y}dy=-\int_{\mathcal{C}_{13}}\frac{\rho_2(y)}{x-y}dy-\int_{\mathcal{C}_{13}}\frac{\rho_1(y)-\rho_2(y)}{x-y}dy$$

where the last term accounts for the mismatch in densities in cuts $\mathcal{C}_{13}$ and is clearly also a $\mathcal{O}(1/L)$ effect. Bellow we will compute this mismatch and find

$$\rho_1(x)-\rho_2(x)=\frac{\Delta\cot_{12}}{2\pi iL}=\frac{\cot_{21}^{+}-\cot_{23}^{+}}{2\pi iL}\quad,\quad x\in\mathcal{C}_{13} \tag{3.2}$$

where $\Delta f\equiv f(x+i0)-f(x-i0)$ and

$$\cot_{ij}\equiv\frac{p_i'-p_j'}{2}\cot\frac{p_i-p_j}{2}\,. \tag{3.3}$$

Thus we find, for $x\in\mathcal{C}_{23}$,

$$\frac{1}{x}+2\,\fint_{\mathcal{C}_{23}}\frac{\rho_2(y)dy}{x-y}+\int_{\mathcal{C}_{13}}\frac{\rho_2(y)dy}{x-y}=2\pi n_{23}^{A}+\phi_2-\phi_3-\frac{1}{L}\left[\cot_{23}-\int_{\mathcal{C}_{13}}\frac{\Delta\cot_{12}}{x-y}\frac{dy}{2\pi i}\right] \tag{3.4}$$

As explained before, if we apply the duality transformation, cuts $\mathcal{C}_{23}$ become cuts $\mathcal{C}_{13}$ and vice-versa and, to leading order, $p_1\leftrightarrow p_2$. Thus for cuts $\mathcal{C}_{13}$ we find precisely the same equation (3.4) with $1\leftrightarrow 2$, so that for $x\in\mathcal{C}_{13}$

$$\frac{1}{x}+2\,\fint_{\mathcal{C}_{13}}\frac{\rho_2(y)dy}{x-y}+\int_{\mathcal{C}_{23}}\frac{\rho_2(y)dy}{x-y}=2\pi n_{13}^{A}+\phi_1-\phi_3-\frac{1}{L}\left[\cot_{13}-\int_{\mathcal{C}_{23}}\frac{\Delta\cot_{12}}{x-y}\frac{dy}{2\pi i}\right] \tag{3.5}$$

These two equations describing the finite size corrections for the two types of cuts of the $su(1,2)$ spin chain are the main results of this section.

In what follows we will derive this result from two different angles. Namely, we will find this finite size corrections using a Baxter like formalism based on transfer matrices for this spin chain in several representations and by exploiting the duality we mentioned in the previous section. It will become clear that the generalization to other NBA equations based on higher rank symmetry groups is straightforward.

[3]recall that the Bethe roots $u_{2,k}$ belongs to a $\mathcal{C}_{23}$ cut and therefore is always well separated from $u_{1,j}$ roots which always belong to $\mathcal{C}_{13}$ cuts.

3.1 Derivation using the transfer matrices

The central object in the study of integrable systems is the *transfer matrix* $\hat{T}(u)$. The algebraic Bethe ansatz formalism has the diagonalization of such objects as main goal and the Bethe equations appear in the process of diagonalization (see [41] and references therein for an introduction to the algebraic Bethe ansatz). As functions of a spectral parameter u and of the Bethe roots $u_{a,j}$ these transfer matrices seem to have some poles at the positions of the Bethe roots. On the other hand they are defined as a product of R operators which do not have these singularities. This means that the residues of these apparent poles must vanish. These analyticity conditions (on the Bethe roots) turn out to be precisely the Bethe equations, and thus, if we manage to obtain the eigenvalues of the transfer matrices, we can use this condition of pole cancellation to obtain the Bethe equations without going through the algebraic Bethe ansatz procedure, see for example [42 – 44, 24]. For the $su(1,2)$ spin chain we have the following transfer matrices in the anti-symmetric representations:

$$T_{\square}(u) = e^{-i\phi_2}\frac{Q_1(u-\frac{3i}{4})}{Q_1(u+\frac{i}{4})}\frac{Q_2(u+\frac{3i}{4})}{Q_2(u-\frac{i}{4})}\left(\frac{u-\frac{5i}{4}}{u-\frac{3i}{4}}\right)^L \tag{3.6}$$

$$+e^{-i\phi_1}\frac{Q_1(u+\frac{5i}{4})}{Q_1(u+\frac{i}{4})}\left(\frac{u-\frac{5i}{4}}{u-\frac{3i}{4}}\right)^L + e^{-i\phi_3}\frac{Q_2(u-\frac{5i}{4})}{Q_2(u-\frac{i}{4})}\left(\frac{u-\frac{5i}{4}}{u+\frac{i}{4}}\right)^L ,$$

$$T_{\substack{\square\\\square}}(u) = \bar{T}_{\square}(\bar{u})\left(\frac{u-\frac{5i}{4}}{u+\frac{5i}{4}}\right)^L \quad , \quad T_{\substack{\square\\\square\\\square}}(u) = \left(\frac{u-\frac{5i}{4}}{u+\frac{5i}{4}}\right)^L . \tag{3.7}$$

One can easily see that the Bethe equations do follow from requiring analyticity of these transfer matrices.

In [16] it was shown and emphasized that the TQ relations are the most powerful method to extract finite size corrections to the scaling limit of Bethe equations.

In this section we will use the transfer matrices presented above along with the fact that, due to the Bethe equations, they are good analytical functions of u to find what are the finite size corrections to this Nested Bethe ansatz. Since for generic (super) nested Bethe ansatz the transfer matrices in the several representations are known, this procedure can be easily generalized for other NBA's.

The key idea to find the finite size corrections to NBA is to use the transfer matrices in the various representations to define a new set of quasi-momenta q_i as the solutions of an algebraic equation whose coefficients are these transfer matrices. For example, to leading order,

$$\begin{aligned} T_{\square}(u) &\simeq e^{ip_1}+e^{ip_2}+e^{ip_3} , \\ T_{\substack{\square\\\square}}(u) &\simeq e^{i(p_1+p_2)}+e^{i(p_2+p_3)}+e^{i(p_3+p_1)} , \\ T_{\substack{\square\\\square\\\square}}(u) &\simeq e^{i(p_1+p_2+p_3)} , \end{aligned}$$

so that if we define a set of *exact* quasimomenta q_i by[4]

$$T_{\substack{\square\\\square\\\square}}(u) - e^{iq}\,T_{\substack{\square\\\square}}(u)\left(1-\frac{L}{4u^2}\right) + e^{2iq}\,T_{\square}(u)\left(1-\frac{L}{4u^2}\right) - e^{3iq} = 0 , \tag{3.8}$$

[4]Exploiting the similarity between this definition equation and 4.1 in [24] we can easily generalize this

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then, to leading order, $q_i \simeq p_i$. Notice however that the coefficients in this equation have no singularities except some fixed poles close to $u = 0$. Thus, defined in this way, the quasi-momenta q_i constitute a 4 sheet algebraic surface (modulo 2π ambiguities) such that

$$\not{q}_i - \not{q}_j = 2\pi n_{ij}^A \ , \ \ x \in \mathcal{C}_{ij} \,, \tag{3.9}$$

and, needless to say, this is an *exact* result in L, it is not a classical (scaling limit) leading result like (2.13)! On the other hand, the expansion at large L of the above algebraic equation yields

$$\begin{aligned} q_1 &= p_1 + \frac{1}{2L}(+\cot_{12} + \cot_{13}) \\ q_2 &= p_2 + \frac{1}{2L}(-\cot_{21} + \cot_{23}) \\ q_3 &= p_3 + \frac{1}{2L}(-\cot_{31} - \cot_{32}) \,, \end{aligned}$$

which follows from the expansion

$$\begin{aligned} T_{\square}(u)\left(1-\frac{L}{4u^2}\right) &= e^{ip_1}+e^{ip_2}+e^{ip_3} \\ &\quad -\frac{1}{4L}\left[e^{ip_1}(2p_1'-p_2'-p_3')+e^{ip_2}(p_1'-p_3')+e^{ip_3}(p_1'+p_2'-2p_3')\right]+\mathcal{O}\left(\frac{1}{L^2}\right) \\ T_{\square\!\square}(u)\left(1-\frac{L}{4u^2}\right) &= e^{i(p_1+p_2)}+e^{i(p_2+p_3)}+e^{i(p_3+p_1)}-\frac{1}{4L}\left[e^{i(p_1+p_2)}(p_1'+p_2'-2p_3')\right. \\ &\quad \left.+e^{i(p_1+p_3)}(p_1'-p_3')+e^{i(p_2+p_3)}(2p_1'-p_2'-p_3')\right]+\mathcal{O}\left(\frac{1}{L^2}\right), \\ T_{\square\!\square\!\square}(u) &= e^{i(p_1+p_2+p_3)}+\mathcal{O}\left(\frac{1}{L^2}\right) . \end{aligned}$$

of the several transfer matrices. Then, to the first order in $1/L$ the exact equation (3.9) gives, for the quasi-momenta p_i introduced in (2.9),

$$\not{p}_2 - \not{p}_3 = 2\pi n_{23}^A - \frac{1}{L}\cot_{23} \ , \qquad x \in \mathcal{C}_{23} \tag{3.10}$$

$$\not{p}_1 - \not{p}_3 = 2\pi n_{13}^A - \frac{1}{2L}\left(\cot_{12} + 2\cot_{13} + \cot_{32}\right) \ , \qquad x \in \mathcal{C}_{13} \tag{3.11}$$

where in (3.10) we use the fact that function $\cot_{31} - \cot_{21}$ vanishes under the slash on the cut $\mathcal{C}_{23}$ since

$$\cot_{ij}^{+} = \cot_{kj}^{-} \ , \ \ x \in \mathcal{C}_{ik} \,. \tag{3.12}$$

Equations (3.10), (3.11) are the finite size corrections we aimed at!

Finally q_2 must have no discontinuity at a cut $\mathcal{C}_{13}$ and therefore

$$\Delta p_2 = 2\pi i\,(\rho_1 - \rho_2) = \frac{1}{L}(\cot_{21}^{+} - \cot_{23}^{+}) \ , \ \ x \in \mathcal{C}_{13} \,. \tag{3.13}$$

algebraic equation to a more general $su(K|M)$ super group. More precisely we identify $e^{2\partial_u} \leftrightarrow e^{iq}$ which is obviously natural if we look at 4.2 in this same paper (see also appendix A where we use this two expressions slightly modified to match our normalizations). We thanks V.Kazakov for pointing this out to us.

Thus, replacing the quasi-momenta p_i by its expressions in terms of resolvents (2.9) and relating the density of *auxiliary roots* ρ_1 to that of the *middle node* roots ρ_2 through (3.13), we recover precisely (3.4) and (3.5) as announced.

We would like to stress the efficiency of the TQ relations. We were able to find the *usual* cot contributions (coming from the expansion of the log's of the Bethe equations when the Bethe roots are close to each other) plus the mismatch in densities of the different types of roots making the cuts of stacks using only the fact that due to Bethe equations the transfer matrices in several representations were analytical functions of u. The computation done in this way is by far more economical than a brute force expansion of the Bethe equations.

Finally let us make an important remark. To derive (3.5) from (3.11) one should use

$$\cot_{12} = -\frac{1}{2\pi i} \int_{\mathcal{C}_{13}\cup\mathcal{C}_{23}} \frac{\Delta \cot_{12}}{x-y} dy \tag{3.14}$$

which is clearly a valid relation if $\cot_{12}$ has only branch cuts as singularities. For generic twists and for small enough cuts $\mathcal{C}_{13}$ and $\mathcal{C}_{23}$ this is the case. Indeed, in the absence of Bethe roots we have no cuts at all and thus $p_1 - p_2 = \phi_2 - \phi_1$. Suppose $\phi_2 - \phi_1 \neq 2\pi n$. Then, by continuity, when we slowly open some cuts $\mathcal{C}_{23}$ and $\mathcal{C}_{13}$ then $p_1 - p_2$ will start taking positive values around $\phi_2 - \phi_1$ without ever being zero. Thus, if the cuts are small enough we will never get poles in $\cot_{12}$. In the next section we will see that the stacks as described in [9] only exist when this assumption of absence of poles is right and are destroyed when $p_1 - p_2$ reaches $2\pi n$.

3.2 Re-derivation using the bosonic duality in the scaling limit

In this section let us re-derive the mismatch formula (3.2) using the bosonic duality (5.1). Besides the obvious advantage for what concerns our comprehension of having a second derivation there are systems for which the Bethe equations are known but the algebraic formalism behind these equations is still not well developed (this is the case for example for the AdS/CFT Bethe equations proposed by Beisert and Staudacher which we will study in section 6).

Denoting

$$u_{1,i} = u_{2,i} - \epsilon_i \;\;,\;\; \tilde{u}_{1,i} = u_{2,i} - \tilde{\epsilon}_i \;\;,\;\; \epsilon \sim 1$$

and expanding the bosonic duality (5.1) in the scaling limit ($L \to \infty$) we get

$$\sin(\tau/2) = \sin\left(\frac{1}{2}\left(\tilde{G}_1 - G_1 + \tau\right)\right) \exp\left(\sum_{i=1}^{K_1} \frac{\epsilon_i}{u - u_i^1} + \sum_{i=1}^{\tilde{K}_1} \frac{\tilde{\epsilon}_i}{u - u_i^1}\right) ,$$

where $\tau = \phi_1 - \phi_2$. Taking the logarithm of this equation and differentiating with respect to u we get

$$\sum \frac{\epsilon_i}{(u-u_i^1)^2} + \sum \frac{\tilde{\epsilon}_i}{(u-u_i^1)^2} = \frac{\tilde{G}_1' - G_1'}{2L} \cot \frac{\tilde{G}_1 - G_1 + \tau}{2}$$

where we notice that the left hand side is precisely the difference of resolvents $G_2 - G_1 - \tilde{G}_1$! Thus we find

$$G_2 - G_1 - \tilde{G}_1 = \frac{\tilde{G}_1' - G_1'}{2L} \cot \frac{\tilde{G}_1 - G_1 + \tau}{2} \simeq \frac{G_2' - 2G_1'}{2L} \cot \frac{G_2 - 2G_1 + \tau}{2} = \frac{1}{L} \cot_{12} \,.$$

Finally, by computing the discontinuity of this expression at the cuts $\mathcal{C}_{13}$ we will get the *mismatch* of the densities of the roots in a cut of stacks[5]

$$\rho_1 - \rho_2 = \frac{\Delta \cot_{12}}{2\pi i L} = \frac{\cot_{21}^+ - \cot_{23}^+}{2\pi i L} \,,$$

which was the gap in the chain of arguments presented in the beginning of the section 3 and leading to (3.4).

Finally let us show that the bosonic duality amounts to a simple exchange of Riemann sheets in the scaling limit. Consider for example

$$\tilde{p}_1 = -\frac{1}{2x} + \tilde{G}_1 - \tilde{\phi}_1 = -\frac{1}{2x} + G_2 - G_1 - \tilde{\phi}_1 = p_2$$

since, as we will see more carefully in the next section, $\tilde{\phi}_{1,2} = \phi_{2,1}$.

4. 1-loop shift

In [31] we explained how to obtain the spectrum of the fluctuation energies around any classical string solution using the algebraic curve by adding a pole to this curve. In particular we reproduced in this way some previous results [45 – 48] where the semi-classical quantization around some simple circular string motions were considered by directly expanding the Metsaev-Tseytlin action [49] around some classical solutions and quantizing the resulting quadratic action. Using the fact that one extra pole in the algebraic curve means one quantum fluctuation, we can compute the leading quantum corrections to the classical energy of the state from the field theory considerations using the algebraic curve alone, as we mentioned in the introduction. This implies a nontrivial relation between fluctuations on algebraic curve and finite size corrections in Bethe ansatz as we will explain in greater detail below. In this section we study this relation on the example of the $su(1,2)$ spin chain and then in section 6 we extend this to the super-string case.

As mentioned in the introduction, in the scaling limit $u \sim K \sim L \gg 1$ we are describing some slow and low energetic spin waves,

$$\mathcal{E} = \sum_{j=1}^{K} \epsilon_j = \sum_{j=1}^{K} \frac{1}{u_{2,j} + 1/4} \sim 1/L \,,$$

around the ferromagnetic vacuum of the theory. In this limit the theory is well described by a Landau-Lifshistz model which is a field theory with coupling $1/L$ [50 – 52]. Therefore a very nontrivial property relating fluctuations and finite size corrections in this NBA should hold:

[5] $\Delta f = f^+ - f^-$, so that $\rho = -\frac{\Delta G}{2\pi i}$

- Suppose we compute the energy shift $\delta\mathcal{E}_n^{ij}$ due to the addition of a stack with mode number n uniting sheets p_i and p_j to a given configuration with some finite cuts $\mathcal{C}$.
- Suppose on the other hand that we compute $1/L$ energy expansion $\mathcal{E} = \mathcal{E}^{(0)} + \frac{1}{L}\mathcal{E}^{(1)} + \dots$ of the configuration with the finite cuts $\mathcal{C}$.

From the field theory point of view the first quantity corresponds to *one of the fluctuation energies* around a classical solution parameterized by the configuration with the cuts $\mathcal{C}$ whereas the second quantity, $\mathcal{E}^{(1)}$, is the 1-*loop shift* [53] around this classical solution with energy $\mathcal{E}^{(0)}$. This 1-loop shift, or ground state energy, is given by the sum of halves of the fluctuation energies [53]

$$\mathcal{E}^{(1)} = \frac{1}{2}\sum_n \sum_{ij} \delta\mathcal{E}_n^{ij} \tag{4.1}$$

In fact for usual (non super-symmetric) field theories this sum is divergent and needs to be regularized. We will see that (4.1) can be generalized and holds for arbitrary local charges

$$\mathcal{Q}_r^{(1)} = \frac{1}{2}\sum_n \sum_{ij} \delta\mathcal{Q}_{r,n}^{ij} \,. \tag{4.2}$$

Let us stress once more that from the Bethe ansatz point of view these quantities are computed independently and there is a priori no obvious reason why such relation between fluctuations and finite size corrections should hold. In this section we will show that Nested Bethe Ansatz's describing (super) spin chains with arbitrary rank do indeed obey such property with some particular regularization procedure (for the Heisenberg $su(2)$ spin chain a similar treatment was carried in [15]). Moreover we will see that the regularization mentioned above also appears naturally from the Bethe ansatz point of view as some integrals around the origin.

4.1 1-loop shift and fluctuations

In this section we will understand the interplay between fluctuations and finite size corrections in NBA's in the scaling limit. For simplicity we are considering the $su(1,2)$ spin chain described in the previous sections. General $su(N)$ is considered in appendix B.

Let us pick the leading order integral equation for the densities of the Bethe roots in the scaling limit (3.1) and perturb it by a single stack, connecting $p^{\rm i}$ with $p^{\rm j}$. According to (2.8) this means simply implies $\rho_2 \to \rho_2 + \frac{1}{L}\delta(x - x^{\rm ij})$, where $x^{\rm ij}$ is position of the new stack. Finally, the positions where one can put an extra stack, as it follows from the BAE (2.3), (2.4), can be parametrized by one integer mod number n

$$p_{\rm i}(x_n^{\rm ij}) - p_{\rm j}(x_n^{\rm ij}) = 2\pi n \,. \tag{4.3}$$

Therefore, for ${\rm i} = 2,\ {\rm j} = 3$ the perturbed equation (3.1) reads

$$\frac{1}{x} + 2 \fint_{\mathcal{C}_{23}} \frac{\rho(y)}{x-y} + \int_{\mathcal{C}_{13}} \frac{\rho(y)}{x-y} + \frac{1}{L}\frac{2}{x - x_n^{23}} = 2\pi k_{23} + \phi_2 - \phi_3 \ \ , \ \ x \in \mathcal{C}_{23} \,. \tag{4.4}$$

and this perturbation will lead to some perturbation of the density $\delta\rho(y)$, which will lead to the perturbation in the local charges (2.5) as

$$\delta \mathcal{Q}_{r,n}^{23} = \int \frac{\delta\rho(y)}{y^r} dy + \frac{1}{L(x_n^{23})^r}\,, \tag{4.5}$$

the local charges of the fluctuation with polarization 23 and mode number n.

Thus, by linearity, if we want to obtain the 1-loop shift (4.2) (or rather a large N regularized version of this quantity where the sum over n goes from $-N$ to N) we have to solve the following integral equation for densities

$$\frac{1}{x} + 2 \fint_{\mathcal{C}_{23}} \frac{\rho(y)}{x-y} + \int_{\mathcal{C}_{13}} \frac{\rho(y)}{x-y} + \sum_{n=-N}^{N} \frac{1}{2L}\left[\frac{2}{x-x_n^{23}} + \frac{1}{x-x_n^{13}}\right] = 2\pi k_{23} \ \ , \ \ x \in \mathcal{C}_{23}\,. \tag{4.6}$$

and then the 1-loop shifted charges are given

$$\mathcal{Q}_r = \int_{\mathcal{C}_{13}\cup\mathcal{C}_{23}} \frac{\rho(y)}{y^r} dy + \sum_{n=-N}^{N} \frac{1}{2L}\left[\frac{1}{(x_n^{23})^r} + \frac{1}{(x_n^{13})^r}\right] \tag{4.7}$$

$$= \int_{\mathcal{C}_{13}\cup\mathcal{C}_{23}} \frac{\rho(y)}{y^r} dy + \sum_{n=-N}^{N} \frac{1}{2L}\left[\oint_{x_n^{23}} \frac{\cot_{23}}{y^r}\frac{dy}{2\pi i} + \oint_{x_n^{13}} \frac{\cot_{13}}{y^r}\frac{dy}{2\pi i}\right]. \tag{4.8}$$

To pass from the first line to the second in the above expression we use that $\cot_{\rm ij}$ has poles at $x_n^{\rm ij}$ with unit residue. We will now understand how to redefine the density in such a way that the second term is absorbed into the first one. We start by opening the contours in (4.8) around the excitation points x_n^{ij}. These contours will then end up around the cuts $\mathcal{C}_{kl}$ of the classical solution and around the origin. We will not consider the contour around $x=0$ — this contribution would lead to a regularization of the divergent sum in r.h.s. of (4.2). We will analyze it carefully in the super-string case, where it leads to the Hernandez-Lopez phase factor. Then we get

$$\mathcal{Q}_r = \int_{\mathcal{C}_{13}\cup\mathcal{C}_{23}} \frac{\rho(y)}{y^r} dy + \frac{1}{2L}\left[\oint_{\mathcal{C}_{13}} \frac{\cot_{23}}{y^r}\frac{dy}{2\pi i} + \oint_{\mathcal{C}_{23}} \frac{\cot_{13}}{y^r}\frac{dy}{2\pi i}\right] \tag{4.9}$$

Noting that

$$\cot_{ij}^{+} = \cot_{kj}^{-} \ \ , \ \ x \in \mathcal{C}_{ik}\,, \tag{4.10}$$

where the superscript $+$ $(-)$ indicates that x is slightly above (below) the cut, we can write

$$\mathcal{Q}_r = \int_{\mathcal{C}_{13}\cup\mathcal{C}_{23}} \frac{\rho(y)}{y^r} dy - \frac{1}{2L}\int_{\mathcal{C}_{13}\cup\mathcal{C}_{23}} \frac{\Delta\cot_{12}}{y^r}\frac{dy}{2\pi i} \tag{4.11}$$

so that we see that it is natural to introduce a new density, "dressed" by the virtual particles,

$$\varrho = \rho - \frac{1}{2L}\frac{\Delta\cot_{12}}{2\pi i} \tag{4.12}$$

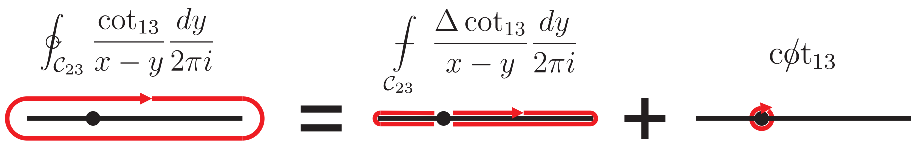

Figure 5: Illustration of an identity used in the main text.

so that the expression for the local charges takes the standard form

$$\mathcal{Q}_r = \int\limits_{\mathcal{C}_{13}\cup\mathcal{C}_{23}} \frac{\varrho(y)}{y^r}dy\,.$$

Let us now rewrite our original integral equation (4.6) in terms of this dressed density. We will see that the integral equation we are constructing for this density by requiring a proper semi-classical quantization will be precisely the equation (3.4) which is the finite size corrected integral equation arising from the NBA for the spin chain! This will thus prove the announced property relating finite size corrections and 1-loop shift. Consider for example the first summand in (4.6) (recall that $x \in \mathcal{C}_{23}$),

$$\sum_n \frac{1}{x-x_n^{23}} = \sum_n \oint_{x_n^{23}} \frac{\cot_{23}}{x-y}\frac{dy}{2\pi i} = \cot_{23} + \oint_{\mathcal{C}_{13}} \frac{\cot_{23}}{x-y}\frac{dy}{2\pi i} = \cot_{23} - \int_{\mathcal{C}_{13}} \frac{\Delta\cot_{12}}{x-y}\frac{dy}{2\pi i}\,, \tag{4.13}$$

Note that $\cot_{23}$ has branch cut singularities at $\mathcal{C}_{13}$ which we have to encircle when we blow up the contour, which leads to the second term. The first term comes from the pole at $x = y$. Finally, to write the second term as it is we used (4.10). Analogously (see figure 5 for a pictorial explanation of the second equality)

$$\sum_n \frac{1}{x-x_n^{13}} = \oint_{\mathcal{C}_{23}} \frac{\cot_{13}}{x-y}\frac{dy}{2\pi i} = \mathrm{c\not{o}t}_{13} + -\!\!\!\!\!\!\int\limits_{\mathcal{C}_{23}} \frac{\Delta\cot_{13}}{x-y}\frac{dy}{2\pi i} = \mathrm{c\not{o}t}_{13} - -\!\!\!\!\!\!\int\limits_{\mathcal{C}_{23}} \frac{\Delta\cot_{12}}{x-y}\frac{dy}{2\pi i}\,. \tag{4.14}$$

Then we note that (see (3.14))

$$\mathrm{c\not{o}t}_{13} = \mathrm{c\not{o}t}_{12} = - -\!\!\!\!\!\!\int\limits_{\mathcal{C}_{13}\cup\mathcal{C}_{23}} \frac{\Delta\cot_{12}}{x-y}\frac{dy}{2\pi i}$$

so that (4.6) reads

$$\frac{1}{x}+2-\!\!\!\!\!\!\int\limits_{\mathcal{C}_{23}}\frac{\rho(y)}{x-y}+\int\limits_{\mathcal{C}_{13}}\frac{\rho(y)}{x-y}+\frac{1}{2L}\left[2\cot_{23}-2-\!\!\!\!\!\!\int\limits_{\mathcal{C}_{23}}\frac{\Delta\cot_{12}}{x-y}\frac{dy}{2\pi i}-3\int\limits_{\mathcal{C}_{13}}\frac{\Delta\cot_{12}}{x-y}\frac{dy}{2\pi i}\right]=2\pi k_{23}+\phi_2-\phi_3$$

which in terms of the redefined density ϱ becomes

$$\frac{1}{x}+2-\!\!\!\!\!\!\int\limits_{\mathcal{C}_{23}}\frac{\varrho(y)}{x-y}+\int\limits_{\mathcal{C}_{13}}\frac{\varrho(y)}{x-y}+\frac{1}{L}\left[\cot_{23}-\int\limits_{\mathcal{C}_{13}}\frac{\Delta\cot_{12}}{x-y}\frac{dy}{2\pi i}\right]=2\pi k_{23}+\phi_2-\phi_3$$

which coincides precisely with (3.4) as announced above! Thus the finite size corrections to the charge of any given configuration will indeed be equal to the field theoretical prediction, that is to the 1-loop shift around the classical solution.

5. Bosonic duality

In this section we will explain some details behind the bosonic duality[6] (2.12) mentioned in section 2. There are two main steps to be considered. On the one hand we have to prove that for a set of K_2 generic complex numbers u_2 and K_1 roots u_1 obeying the auxiliary Bethe equations (2.3) it is possible to write ($\tau = \phi_1 - \phi_2$)

$$2i \sin(\tau/2)\, Q_2(u) = e^{i\tau/2} Q_1(u - i/2)\tilde{Q}_1(u + i/2) - e^{-i\tau/2} Q_1(u + i/2)\tilde{Q}_1(u - i/2)\,, \quad (5.1)$$

and that, in doing so, we define the position of a new set of numbers $\tilde{u}_1$. A priori this is not at all a trivial statement because we have a polynomial of degree K_2 on the left whereas on the right hand side we have only $K_2 - K_1$ parameters to fix. However, as we will see, if K_1 equations (2.3) are satisfied it is possible to write $Q_2(u)$ in this form. This will be the subject of the section 5.1.

Assuming (5.1) to be proved we can use this relation to show that in the original Bethe equations we can replace the roots u_1 by the new roots $\tilde{u}_1$ with the simultaneous exchange $\phi_1 \leftrightarrow \phi_2$. Indeed if we evaluate the duality at $u = u_{2,j}$ we find

$$\frac{Q_1(u_{2,j} - i/2)}{Q_1(u_{2,j} + i/2)} = e^{i(\phi_2 - \phi_1)} \frac{\tilde{Q}_1(u_{2,j} - i/2)}{\tilde{Q}_1(u_{2,j} + i/2)}\,,$$

meaning that in the equation (2.4) for the u_2 roots we can replace the roots u_1 by the dual roots $\tilde{u}_1$ provided we replace $\phi_1 \leftrightarrow \phi_2$. Moreover if we take $u = \tilde{u}_{1,j} \pm i/2$ we will get

$$e^{i\phi_2 - i\phi_1} = -\frac{\tilde{Q}_1(\tilde{u}_1 + i)}{\tilde{Q}_1(\tilde{u}_1 - i)} \frac{Q_2(\tilde{u}_1 - i/2)}{Q_2(\tilde{u}_1 + i/2)}\,,$$

which we recognize as equation (2.3) with $K_2 - K_1$ roots $\tilde{u}_1$ in place of the K_1 original roots u_1 and with $\phi_1 \leftrightarrow \phi_2$. Finally evaluating (5.1) at $u = u_{1,j} \pm i/2$ we will get the original equation (2.3) so that we see that it must be satisfied in order to equation (5.1) to be valid.

In section 5.2 we will also see that the transfer matrices are invariant under the bosonic duality accompanied by an appropriate reshuffling of the phases ϕ_a. In section 5.3 some curious examples of dual states will be given.

5.1 Decomposition proof

In this section we shall prove that one can always decompose $Q_2(u)$ as in (5.1) and that this decomposition uniquely fixes the position of the new set of roots $\tilde{u}_1$. In other words, let us show that we can set the polynomial

$$P(u) \equiv e^{+i\frac{\tau}{2}} Q_1(u - i/2)\tilde{Q}_1(u + i/2) - e^{-i\frac{\tau}{2}} Q_1(u + i/2)\tilde{Q}_1(u - i/2) - 2i \sin\frac{\tau}{2} Q_2(u)$$

to zero through a unique choice of the dual roots $\tilde{u}_1$.

[6] Bazhanov and Tsuboi also found some similar duality in the study of the deformed $U_q(sl(1|1))$. We thanks Z.Tsuboi for providing us the talk he gave at the "t9me rencontre entre physiciens theoriciens et mathmaticiens: Supersymmetry and Integrability" (http://www-irma.u-strasbg.fr/article383.html) and V.Kazakov who informed us of their work. It would be very interesting to connect both approaches.

- Consider first the case $K_1 = 0$. Then it is trivial to see that we can always find unique polynomial $\tilde{Q}_1 = u^{K_2} + \sum_{n=1}^{K_2} a_n u^{n-1}$ such that

$$e^{+i\frac{\tau}{2}}\tilde{Q}_1(u+i/2) - e^{-i\frac{\tau}{2}}\tilde{Q}_1(u-i/2) = 2i\sin\frac{\tau}{2}Q_2(u)\,.$$

because this amounts to solving K_2 linear equations for K_2 coefficients a_n with non-degenerate triangular matrix.

- Next let us consider $K_1 \le K_2/2$. First we choose $\tilde{Q}_1$ to satisfy K_1 equations

$$\tilde{Q}_1(u_p^1) = 2ie^{-i\frac{\tau}{2}}\sin\frac{\tau}{2}\frac{Q_2(u_p^1 - i/2)}{Q_1(u_p^1 - i)} \equiv c_p \quad , \quad p = 1, \dots, K_1$$

these conditions will define $\tilde{Q}_1(u)$ up to a homogeneous solution proportional to $Q_1(u)$,

$$\tilde{Q}_1(u) = Q_1(u)\tilde{q}_1(u) + \sum_{p=1}^{K_1}\frac{Q_1(u)}{Q_1'(u_p^1)(u - u_p^1)}\,c_p$$

where $\tilde{q}_1(u)$ is some polynomial of the degree $K_2 - 2K_1$. Now from (2.3) we notice that with this choice of $\tilde{Q}_1$ we have

$$\frac{P(u_p^1 + i/2)}{Q_2(u_p^1 + i/2)} = \frac{P(u_p^1 - i/2)}{Q_2(u_p^1 - i/2)} = 0 \quad , \quad p = 1, \dots, K_3$$

and thus

$$P(u) = Q_1(u+i/2)Q_1(u-i/2)p(u)$$

where

$$p(u) = e^{i\frac{\tau}{2}}\tilde{q}_1(u+i/2) - e^{-i\frac{\tau}{2}}\tilde{q}_1(u-i/2) - 2i\sin\frac{\tau}{2}\,q_2(u)$$

and q_2 is a polynomial. Thus we are left to the same problem as above where $K_1 = 0$. For completeness let us note that we can write $q_2(u)$ explicitly in terms of the original roots u_1 and u_2,

$$q_2(u) = \frac{Q_2(u)}{Q_1(u+i/2)Q_1(u-i/2)} - \text{poles}$$

where the last term is a simple collection of poles at $u = u_p^1 \pm i/2$ whose residues are such that $q_2(u)$ is indeed a polynomial.

- We can see that the number of the solutions of (2.3) with $K_1 = K$ and $K_1 = K_2 - K$ is the same (see [41] for examples of states counting). Thus for each solution with $K_1 \ge K_2/2$ we can always find one dual solution with $K_1 \le K_2/2$ and in this way we prove our statement for $K_1 \ge K_2/2$

- Finally let us stress the uniqueness of the $\tilde{Q}_1$. If $K_1 > \tilde{K}_1$ we have nothing to show since we saw explicitly above how the bosonic duality constrains uniquely the dual polynomial $\tilde{Q}_1$. Let us then consider $K_1 < \tilde{K}_1$ and assume we have two different solutions $\tilde{Q}_1^1$ and $\tilde{Q}_1^2$. Then from the duality relation (5.1) for either solution we find

$$e^{i\frac{\tau}{2}}Q_1(u-i/2)\Big(\tilde{Q}_1^1(u+i/2) - \tilde{Q}_1^2(u+i/2)\Big) = e^{-i\frac{\tau}{2}}Q_1(u+i/2)\Big(\tilde{Q}_1^1(u-i/2) - \tilde{Q}_1^2(u-i/2)\Big).$$

Evaluating this expression at $u = u_{1,j} + i/2$ we find that $\tilde{Q}_1^1(u_{1,j}) - \tilde{Q}_1^2(u_{1,j}) = 0$ so that $\tilde{Q}_1^1(u_1) - \tilde{Q}_1^2(u_1) = Q_1(u)h(u)$ and therefore

$$e^{i\frac{\tau}{2}} h(u + i/2) = e^{-i\frac{\tau}{2}} h(u - i/2)$$

which is clearly impossible for polynomial $h(u)$ — for large u we can neglect the $i/2$'s to obtain $e^{i\tau} = 1$ thus leading to a contradiction.

5.2 Transfer matrix invariance under the bosonic duality

In this section we will examine the transformation properties of the transfer matrices under the bosonic duality. In appendix A we consider this problem for the general $su(N|M)$ group. For now let us just take $T_{\square}$ for $su(1,2)$ from (3.6). Using (5.1) we can express ratios of Q_1's through $\tilde{Q}_1$ and Q_2 so that

$$\begin{aligned} T_{\square}(u) = e^{-i\phi_2} & \left(+ \frac{2i \sin\frac{\tau}{2} e^{-i\frac{\tau}{2}} Q_2(u - \frac{i}{4})}{Q_1(u + \frac{i}{4}) \tilde{Q}_1(u + \frac{i}{4})} + e^{-i\tau} \frac{\tilde{Q}_1(u - \frac{3i}{4})}{\tilde{Q}_1(u + \frac{i}{4})} \right) \frac{Q_2(u + \frac{3i}{4})}{Q_2(u - \frac{i}{4})} \left(\frac{u - \frac{5i}{4}}{u - \frac{3i}{4}} \right)^L \\ & + e^{-i\phi_1} \left(- \frac{2i \sin\frac{\tau}{2} e^{+i\frac{\tau}{2}} Q_2(u + \frac{3i}{4})}{Q_1(u + \frac{i}{4}) \tilde{Q}_1(u + \frac{i}{4})} + e^{+i\tau} \frac{\tilde{Q}_1(u + \frac{5i}{4})}{\tilde{Q}_1(u + \frac{i}{4})} \right) \left(\frac{u - \frac{5i}{4}}{u - \frac{3i}{4}} \right)^L \\ & + e^{-i\phi_3} \frac{Q_2(u - \frac{5i}{4})}{Q_2(u - \frac{i}{4})} \left(\frac{u - \frac{5i}{4}}{u + \frac{i}{4}} \right)^L . \end{aligned}$$

We see that for $\tau = \phi_1 - \phi_2$ the terms with $\sin\frac{\tau}{2}$ cancel and we get the old expression for $T_{\square}$ with u_1 replaced by $\tilde{u}_1$ and $\phi_1 \leftrightarrow \phi_2$.

This simple transformation property of the transfer matrices automatically implies that the Riemann surface defined by the algebraic equation (3.8) is untouched under the duality transformation (to all orders in L), so that the duality can cause at most some reshuffling of the sheets. However, as we will see in the next section, not necessarily the sheets as a whole are exchanged — this operation will be in general done in a piecewise manner.

5.3 Examples

In this section we will study some curious Bethe roots distributions for the twisted $su(1,2)$ spin chain described by the nested Bethe equations (2.3) and (2.4) and for the usual $su(2)$ Heisenberg chain,

$$\left(\frac{u_{1,j} + \frac{i}{2}}{u_{1,j} - \frac{i}{2}} \right)^L = - \frac{Q_1(u_{1,j} + i)}{Q_1(u_{1,j} - i)} . \quad (5.2)$$

Using the first example we shall understand the importance of twists to stabilize big cuts of stacks like the ones depicted in figures $2a$, $2b$ and explain how the stacks gets destroyed as we decrease the twists.

We can dualize $su(2)$ solutions of the twisted[7] Heisenberg ring using the same duality (2.12) as before with $Q_2(u) \to u^L$. We will consider the dual solutions to the vacuum and to a 1-cut solution for the Heisenberg spin chain (5.2) as a prototype of the curious solutions one would get.

[7]For zero twist the duality becomes degenerate and we will see below that it needs to be slightly modified.

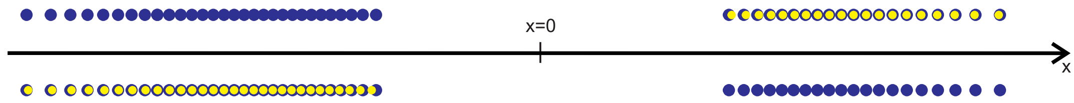

Figure 6: The upper and the lower configuration of Bethe roots are dual to one another. Big blue dots are middle node roots u_2, yellow dots are auxiliary roots u_1. The formation of cuts of stacks is manifest for this situation where the twists are large (like $\pi/2$) and the filling fractions are small.

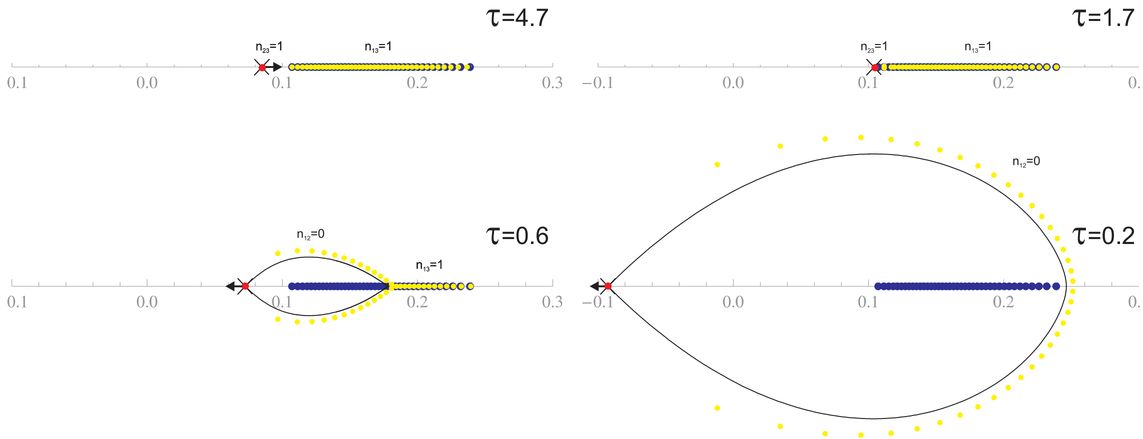

Figure 7: Disintegration of the stack configuration. When the twist is large (the top left corner) the auxiliary roots form bound states together with the middle node ones and constitute a cut of stacks. As we decrease the twist fluctuation $n_{23} = 1$ (the red crossed dot) enters the cut of stacks (the top right corner) and subsequently partly *disintegrate* the cut of stacks forming some zipper like configuration (the bottom left corner). At some very small value of the twist the configuration of Bethe roots bears no resemblance with a cut of stacks.

5.3.1 Big enough twists, small enough fillings and *zippers*

In the previous sections we saw that the introduction of twists in the NBA equations are needed to have a configuration with auxiliary roots u_1 close to some momentum carrying roots u_2. In figure 6 we have two numerical solutions of the Bethe equations which are related by the bosonic duality. In either of them we see a configuration of Bethe roots with a simple cut with middle roots only (in blue) and a cut of stacks (containing blue and yellow roots). In this situation it is clearly reasonable to think of stacks as bound states of different types of roots and we see that they indeed condense into *multicolor* cuts.

We will examine what happens when we decrease the twists (or increase filling fractions, which is the same qualitatively). For simplicity we consider the configuration, dual to the simple one cut solution ($K_2 = K$ and $K_1 = 0$) with no twist for the middle node roots, $\phi_2 - \phi_3 = 0$, and some generic twist $\phi_1 - \phi_2 = \tau$ for the auxiliary roots. Bosonic duality will leave untouched middle node roots u_2 and create K new axillary roots u_1.

In the upper left corner of figure 7 we applied the duality for some big twist $\tau = 4.6$ while in the bottom right corner of the same figure we have a configuration of Bethe roots with some small twist $\tau = 0.2$. In this latter case the auxiliary (yellow) roots clearly do *not* form stacks together with the middle node (blue) roots!, rather they form a bubble,

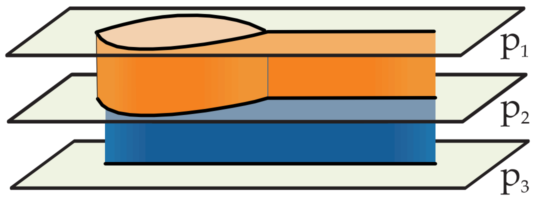

Figure 8: In the scaling limit the algebraic curves for e^{ip_j} are the same before the duality (blue cut only) and after the duality (when the auxiliary roots are created). The duality causes interchange of the sheets outside the bubble, while keeping the order untouched inside. This follows from the need of a positive density for the "virtual" cut. In other words the duality is indeed only interchanging the sheets of the Riemann surface although it is interchanging them in a piecewise way.

containing the original cut of roots u_2.

To understand what happens in the scaling limit consider the position of $n_{23} = 1$ fluctuation, given by (4.3), which would be a small infinitesimal cut between p_2 and p_3. Clearly this probe cut would have no influence on the leading order algebraic curve for p_i. In figure 7 the position of this virtual fluctuation is marked by a red crossed dot. When the twist is big enough (and filling fraction is small enough) the fluctuation is to the left from the cut. When we start decreasing the twist the fluctuation approaches the cut (upper right picture on fig 7) and at this point we have at the same time

$$p_2(x_n) - p_3(x_n) = 2\pi$$

and

$$p_1(x_n) - p_3(x_n) = 2\pi \ ,$$

which implies $p_1 - p_2 = 0$ so that equation (3.14) becomes wrong at this point. When we continue decreasing the twist the fluctuation passes through the cut and becomes a $n_{12} = 0$ fluctuation. If we think of the fluctuation as being a small cut along the real axis we see that density becomes negative after crossing the cut:

$$0 < \rho_{23}^{\rm fluc} = -\frac{\Delta(p_2 - p_3)}{4\pi i} = -\frac{\Delta(-p_1 - p_2)}{4\pi i} = -\rho_{12}^{\rm fluc}$$

This means that two branch points of the infinitesimal cut should not be connected directly, but rather by some macroscopical curve with real positive density! This curves $z(t)$ can be calculated from the equation $\rho(z)dz \in \mathbb{R}^+$ or

$$\frac{p_1(z) - p_2(z)}{2\pi i}\partial_t z = \pm 1$$

and the resulting curve is plotted in black on the two bottom pictures on the figure 7. This is very similar to what happens when a fluctuation passes through the 1 cut $su(2)$ configuration [54]. In the scaling limit the black curve corresponds to the cut connecting p_1 and p_2 like on the figure 8.

At first sight these figures seem to be defying our previous results. Indeed we checked in the previous section that the transfer matrices themselves are invariant under the bosonic duality. Thus the algebraic curves obtained from (3.8) should be the same after and before duality and thus what one naturally expects is a simple interchange of Riemann sheets $p_1 \leftrightarrow p_2$ under the duality transformation. What really happens is a bit more tricky. The quasimomenta *are indeed* only exchanged but this exchange operation is done in a *piecewise* manner. That is, if we denote the new quasi-momenta by $p_i^{\rm new}$ and the old ones by $p_i^{\rm old}$ and if we denote the bubble in figure 8 by $\mathcal{R}$ then we have

$$p_1^{\rm new} = \begin{cases} p_2^{\rm old} \ , \ \textit{outside}\ \mathcal{R} \\ p_1^{\rm old} \ , \ \textit{inside}\ \mathcal{R} \end{cases} , \ p_2^{\rm new} = \begin{cases} p_1^{\rm old} \ , \ \textit{outside}\ \mathcal{R} \\ p_2^{\rm old} \ , \ \textit{inside}\ \mathcal{R} \end{cases} , \ p_3^{\rm new} = p_3^{\rm old} \tag{5.3}$$

where the border of the region $\mathcal{R}$ can be precisely determined in the scaling limit as explained above.

5.3.2 Dualizing momentum carrying roots

In this section we will consider an example of application of the bosonic duality to the Heisenberg magnet.[8] The duality (2.12) can be applied to the roots u_1 obeying (5.2) provided we replace $Q_2(u) \to u^L$. In fact if we want to consider strictly zero twist we need a new duality because that one is clearly degenerate in this limiting case. The proper modified expression is in this case

$$i(\tilde{K}_1 - K_1)u^L = Q_1(u - i/2)\tilde{Q}_1(u + i/2) - Q_1(u + i/2)\tilde{Q}_1(u - i/2) \tag{5.4}$$

and the number of dual roots is now $L - K_1 + 1$. Contrary to what happened with non-zero twists, here, the dual solution is not unique. Indeed if $\tilde{K}_1 > K_1$ we can as well use

$$\tilde{Q}_1^\alpha \equiv \alpha\, Q_1 + \tilde{Q}_1\,. \tag{5.5}$$

All these solutions, parameterized by the constant α, have the same charges because the transfer matrix is invariant under this transformation — see appendix A. Notice that if initially we have a physical state with $K_1 < L/2$ roots then all dual states (5.5) are unphysical with $\tilde{K}_1 > L/2$ violating the half-filling condition. Still, it is interesting, at the level of Bethe equations, to understand how these solutions look like. First of all let us single out a particular $\tilde{Q}_1$ out of the various solutions to (5.4) so that

$$\tilde{Q}_1^\alpha = u^{\tilde{K}_1} + \sum_{l=0}^{\tilde{K}_1 - 1} c_l^\alpha\, u^l \tag{5.6}$$

becomes well defined through (5.5). We chose $\tilde{Q}_1 = \tilde{Q}_1^0$ to be the dual solution with $c_0^0 = 0$.

Consider for example the vacuum state for which $Q_1 = 1$. Let us first take α to be very large so that we can write

$$\alpha + \tilde{Q}_1^0 \simeq \alpha + (xL)^L\,. \tag{5.7}$$

[8]This section beneficed a lot from the insightful discussions with T. Bargheer and N. Beisert whom we should thank.

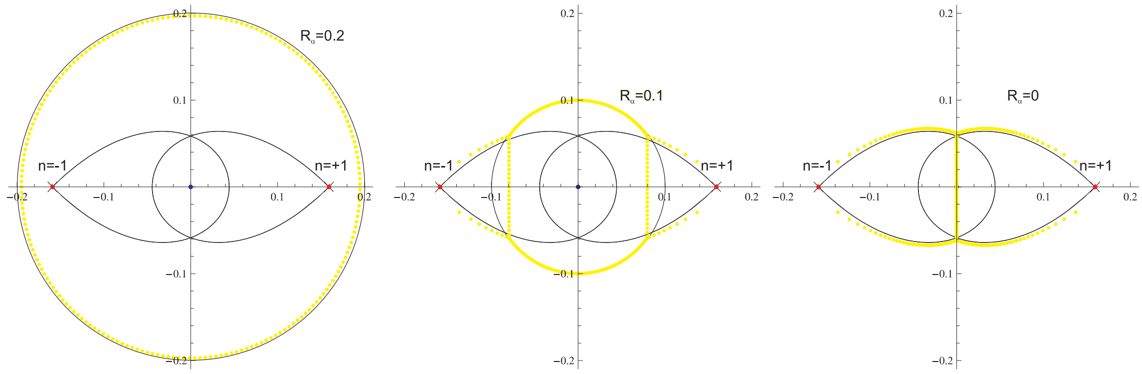

Figure 9: Three configurations of Bethe roots dual to the ferromagnetic vacuum of the untwisted Heisenberg spin chain. For each physical solution (below half filling) of the Bethe equations there is a one parameter (α) family of dual unphysical solutions. To the left, α is large and the roots distribute themselves along a circle with radius R_α given by $(R_\alpha L)^L = \alpha$. Decreasing α the circle will touch the fluctuations $n = \pm 1$. Similarly to the previous section the virtual infinitesimal cuts become macroscopical bubble cuts with cusps at the position of the fluctuations. Intersection points of the new cuts with the circle are connected by condensates, which are logarithmic cuts on the algebraic curve [54].

We see for large α the dual roots will be on a circle of radius $\frac{|\alpha|^{1/L}}{L}$. The corresponding configuration is present on the first picture on the figure 9. In this figure we also plotted a circle with this radius and one can see that the Bethe roots belong perfectly to the circle.

Let us now understand this configuration from the algebraic curve point of view. The the quasi-momenta $p_1 = -p_2 \equiv p = \frac{1}{2x} - G$, in the absence of Bethe roots, are simply given by $p = \frac{1}{2x}$. Let us find the curves with positive densities and mode number $n = 0$. The density is given by $\rho(x) = \frac{1}{2\pi i}\frac{1}{x}$ and we have to find the curves where $\rho(x)dx$ is real. It is easy to see that the only possibility is the circle centered at the origin with an arbitrary radius. From the above arguments one can expect that for any α the roots will belong to some circle. However, we analysed only the curves with zero mode number and as we see on the figure 9 for smaller α's the circle develops four tails and two vertical lines. Along these vertical lines the roots are separated by i (for $L \to \infty$) forming the so called *condensates* or *Bethe strings*. The tails meet at the points where the virtual fluctuation is and the corresponding curves are given by

$$\frac{p(z) \pm \pi}{\pi i}\partial_t z = \pm 1 \tag{5.8}$$

analogously to the previous section. In the last configuration on figure 9 the circle is completely absent. There are only two $n = \pm 1$ curves which, at the interceptions, become a 4π jump log condensate with the Bethe roots separated by $i/2$. We also built the dual configurations to the 1-cut solution (see figure 10). The situation is similar to the vacuum, the only difference being that two tails (out of four) do not tend to touch each other, but rather end at the branch points of the initial cut.

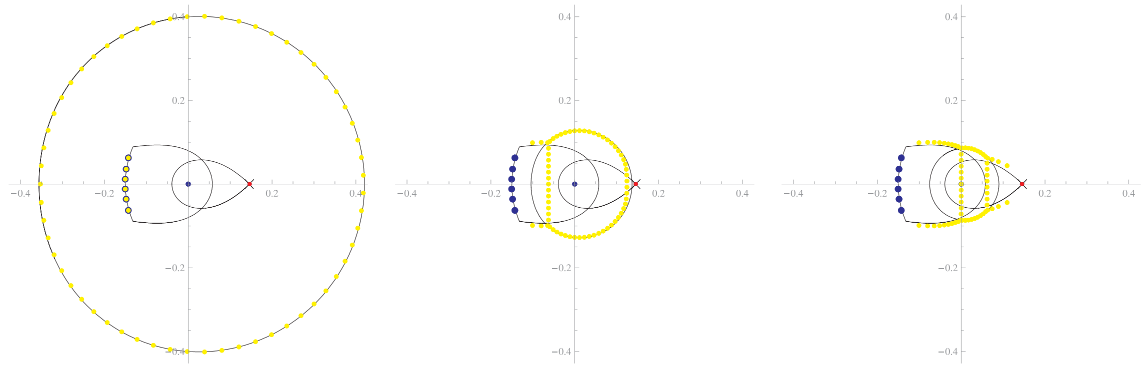

Figure 10: Dual configuration to 1-cut solution. Similar to the previous example for the large α the dual roots are distributed along the big circle and cut (first picture). When the α decreases and the circle crosses the cut we have to choose another curve with the positive density (second and third pictures).

5.4 On twists — partial summary

Let us summarize the main lessons we understood so far concerning the twists we introduced in the Nested Bethe equations (2.1). These twists ϕ_j which proved to be very useful to control the solutions of the Bethe equations.

For example we saw that the simplest nested configuration of Bethe roots is a bound state, called stack, of two different magnons u and v whose positions are given by (2.11). For zero twists this bound state no longer exists since the separation between the roots u and v is proportional to the cotangent of the twists.

The next very important configurations of Bethe roots we needed to consider were cuts of stacks, represented in figures 3, 4. We understood that such configurations can be generated from the usual cuts made out of a single type of Bethe roots via a transformation we called *bosonic duality*. What happens is that a new set of roots $\tilde{u}_1$ appear on top of the roots u_2 of the middle node. In section 3.2 we saw that for this to happen, that is for the new roots to be indeed located close (separated by $\mathcal{O}(1)$) to the roots u_2, we need small enough filling fractions and/or large enough the twists. More precisely, if the cut of stacks will unite the quasimomenta p_1 to p_3 and if the single cuts connects p_2 and p_3 then the densities of $\tilde{u}_1$ and u_2 roots making up the cut of stacks differ by a factor proportional to $\cot(p_1 - p_2)$. The twists ϕ_j ensure that $p_1 - p_2$ never equals an integer multiple of 2π.

Finally, in the previous section some examples of Bethe ansatz configurations are studied both analytically and numerically. These new configurations were generated from simpler ones using the bosonic duality. Since, as explained in section 5.2, the duality leaves the transfer matrices invariant and since the quasimomenta describing the Bethe roots in the scaling limit are the eigenvalues of these matrices, the duality must at most exchange the quasimomenta amongst themselves, $p_a(x) \leftrightarrow p_b(x)$. Indeed we see, as anticipated in the previous sections, that if we start with a cut of simple roots connecting p_2 and p_3 and if we apply the duality then, for large enough twists we do get a cut of stacks connecting p_1 and p_3, the new auxiliary roots being placed close to the preexisting middle node Bethe roots. In this case, upon action of the duality, $p_1(x) \leftrightarrow p_2(x)$. On the other hand we then saw that for small twists something seemingly strange happens. Namely the auxiliary roots start condensing on some closed curves, called *zippers* as depicted in figure 7. In general a complicated configuration of Bethe roots could contain several such closed curves enclosing in this way some isolated regions $\mathcal{R}_i$. As we explain in sections 5.3.1 and 5.3.2 what happens is that the quasimomenta are still interchanged but in a piecewise manner — inside each region $\mathcal{R}_i$ a different permutation of the quasimomenta might (and will) occur. For the purpose of the (semi-) classical curves we can however completely avoid dealing with these regions by simply redefining the quasimomenta. For example in (5.3) we can simply eliminate — that is avoid working with it — the bubble (zipper) by working with the quasimomenta before the duality which are of course equivalent to the quasimomenta after the duality but simply more continuous than these.

Notice also that analytically we expect nothing special to happen for example, all solutions plotted in figure 7 have exactly the same energy since the middle node roots are the same in all cases and only these carry charges!

Summarizing the twists are an important mathematical *tool* to stabilize Bethe equations and to be able to manipulate configuration of Bethe roots which describe classical algebraic curves in the usual straightforward way. On the other hand since no analytical singularities are expected as we decrease the twists the conclusions reached at $\phi_j \neq 0$ can then be analytically continued to the vanishing twist case — see also section 6.7 and [54].

6. The AdS/CFT Bethe equations and the semiclassical quantization of the superstring on $AdS_5 \times S^5$

6.1 Introduction and notation

The Beisert-Staudacher (BS) equations [25] are a set of 7 asymptotic [56] Bethe equations (the rank of the symmetry group PSU$(2,2|4)$) which are expected to describe the anomalous dimensions of $\mathcal{N}=4$ SYM single trace operators with a large number of fields[9] as well as the energy of the dual string states.[10] The perturbative gauge theory and the classical string regimes are interpolated by these equations through the t'Hooft coupling λ. In [57], based on an hypothesis for a natural extension for the quantum symmetry of the theory, Beisert found (up to a scalar factor) an S-matrix from which the BS equations would be derived. The scalar factor was then conjectured in [58, 59] from the string side — using the Janik's crossing relation [60] — and in [61, 62] from the gauge theory point of view — based on several heuristic considerations [63]. From the gauge theory side these equations were tested quite recently up to four loops [64 – 66]. From the string theory point of view the

[9]These large traces can be though of as spin chains and then the dilatation operator behaves like a spin chain Hamiltonian which turns out to be integrable [29, 30]. In this way Bethe equations appear naturally from the gauge theory side.

[10]The existence of a finite gap description of the classical string motion [8, 9] lead to the belief that these equations ought to be the continuous limit of some quantum string Bethe equations. In other words, the Riemann surfaces present therein should in fact be the condensation of a large number of Bethe roots. Inspired by these finite gap constructions these quantum equations were proposed shortly after [55, 25].

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scalar factor recently passed several nontrivial checks [37, 67 − 69] where several loops were probed at strong coupling. Also at strong coupling, the full structure of the BS equations was derived up to two loops in [70, 71] in a particular limit [72] where the sigma model is drastically simplified.

In this section we will check that the BS equations reproduce the 1-loop shift around *any* classical string soliton solution with exponential precision in the large angular momentum in the string state. To do so our computation is divided into two main steps. On the one hand we will compute the $1/\sqrt{\lambda}$ corrections to Bethe equations in the scaling limit. We will have to use the technology developed in the previous sections in order to understand precisely the several sources of corrections, the most subtle of all being the fine structure of the cuts of stacks which are generically present.[11] At the end we will find out some integral equation corrected by a $1/\sqrt{\lambda}$ term.

On the other hand we start from the algebraic curve description of the string classical motion [8, 9]. The integral equations present in this finite gap formalism coincide with the scaling limit of the Bethe equations. Then we find how to correct this equations in such a way that they will now describe not only the classical motion but also the semi-classical quantization of the theory around *any classical motion*. For example we will find out how to modify the equations in such a way that they exhibit a very nontrivial property: the first finite corrections to any classical configurations equals the sum of quantum fluctuations around this same classical configuration. Then we show that, modified in this way, the integral equations coincide precisely with the scaling limit expansion of the BS equations with the HL phase [39] (up to some exponencially supressed wrapping effects, irrelevant for large angular momentum string states)! In this way we establish that, to this order in $1/\sqrt{\lambda}$, the BS equations do provide the correct quantization of the system.

These Bethe equations are a deformation of the equations (2.1) through the introduction of the map

$$x + \frac{1}{x} = \frac{4\pi u}{\sqrt{\lambda}} \ , \ \ x^{\pm} + \frac{1}{x^{\pm}} = \frac{4\pi}{\sqrt{\lambda}} \left(u \pm \frac{i}{2} \right) .$$

As explained in section 2 for superalgebras the choice of Bethe equations is not unique. In [25] four choices are presented. We need only to consider two of them,[12] corresponding to the diagram in figure 1 or to the reflected path along the diagonal going from the lower left to the upper right corner.

Moreover we consider a twisted version of these equations for the same reasons mentioned in the previous sections. In [76, 77] a similar kind of twists were introduced in the study of a set of deformations of $\mathcal{N} = 4$ SYM and of the dual sigma model. Our twists seem to be a simple change in boundary conditions via the introduction of a constant matrix like (2.2). It would be interesting to see if they can also be given a deeper physical interpretation following the lines of these works. We should stress that the twists are used here as a technical tool which will simplify our analysis because, in particular, it allows us to

[11] In [73 − 75] the scaling limit of the SU(3) sector was considered. It would be interesting to use our treatment, including stacks, to compute explicitly the finite size corrections in this subsector following the lines of these papers.

[12] In [25] we consider $\eta_1 = \eta_2 = \eta$.

deal with well defined stacks in a regime where the dualities are nothing but an exchange of Riemann sheets. We will explain in section 6.7 that we can then safely analytically continue the results to zero twist.

The BS equations then read

$$\begin{aligned}
e^{i\eta\phi_1-i\eta\phi_2} &= \prod_{j=1}^{K_2}\frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{1-1/x_{1,k}x^+_{4,j}}{1-1/x_{1,k}x^-_{4,j}}\,,\\
e^{i\eta\phi_2-i\eta\phi_3} &= \prod_{j\neq k}^{K_2}\frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i}\prod_{j=1}^{K_3}\frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}\prod_{j=1}^{K_1}\frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}}\,,\\
e^{i\eta\phi_3-i\eta\phi_4} &= \prod_{j=1}^{K_2}\frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}}\,,\\
e^{i\eta\phi_4-i\eta\phi_5} &= \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^{\eta L}\prod_{j\neq k}^{K_4}\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\prod_{j}^{K_4}\left(\frac{1-1/x^+_{4,k}x^-_{4,j}}{1-1/x^-_{4,k}x^+_{4,j}}\right)^{\eta-1}\left(\sigma^2(x_{4,k},x_{4,j})\right)^{\eta} \quad (6.1)\\
&\quad\times\prod_{j=1}^{K_1}\frac{1-1/x^-_{4,k}x_{1,j}}{1-1/x^+_{4,k}x_{1,j}}\prod_{j=1}^{K_3}\frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}}\prod_{j=1}^{K_5}\frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}}\prod_{j=1}^{K_7}\frac{1-1/x^-_{4,k}x_{7,j}}{1-1/x^+_{4,k}x_{7,j}}\,,\\
e^{i\eta\phi_5-i\eta\phi_6} &= \prod_{j=1}^{K_6}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}}\,,\\
e^{i\eta\phi_6-i\eta\phi_7} &= \prod_{j\neq k}^{K_6}\frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i}\prod_{j=1}^{K_5}\frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\prod_{j=1}^{K_7}\frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}}\,,\\
e^{i\eta\phi_7-i\eta\phi_8} &= \prod_{j=1}^{K_6}\frac{u_{7,k}-u_{6,j}+\frac{i}{2}}{u_{7,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{K_4}\frac{1-1/x_{7,k}x^+_{4,j}}{1-1/x_{7,k}x^-_{4,j}}\,.
\end{aligned}$$

In fact, in order for the fermionic duality [25] (which we will review below) to exist, the twists must not be completely independent but rather

$$\begin{aligned}
\phi_1-\phi_2+\eta\sum_{j=1}^{K_4}\frac{1}{i}\log\frac{x_4^+}{x_4^-} &= \phi_3-\phi_4\,,\\
\phi_7-\phi_8+\eta\sum_{j=1}^{K_4}\frac{1}{i}\log\frac{x_4^+}{x_4^-} &= \phi_5-\phi_6\,. \qquad (6.2)
\end{aligned}$$

The energy (the anomalous dimension) can then be read from

$$\delta D=\frac{\sqrt{\lambda}}{2\pi}\sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\right). \qquad (6.3)$$

To describe classical solutions (and to semi-classically quantize them) we should consider the scaling limit where

$$\sqrt{\lambda}\sim u\sim K_a\sim L\gg 1\,.$$

In this limit we have

$$x^{\pm} = x \pm \frac{i}{2}\,\alpha(x) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

where

$$\alpha(x) \equiv \frac{4\pi}{\sqrt{\lambda}}\frac{x^2}{x^2-1}\,.$$

It is then useful to introduce the resolvents[13]

$$\begin{aligned}
F_a(x) &= \sum_j \frac{1}{u-u_{a,j}}\,, & & \\
G_a(x) &= \sum_j \frac{\alpha(x_{a,j})}{x-x_{a,j}}\,, & \bar{G}_a(x) &= \sum_j \frac{\alpha(1/x_{a,j})}{x-1/x_{a,j}} \\
H_a(x) &= \sum_j \frac{\alpha(x)}{x-x_{a,j}}\,, & \bar{H}_a(x) &= \sum_j \frac{\alpha(1/x)}{1/x-x_{a,j}}
\end{aligned}$$

and build with them eight quasi-momenta ($\mathcal{J} = L/\sqrt{\lambda}$)

$$\begin{aligned}
p_1 &= +\frac{2\pi\mathcal{J}x - \delta_{\eta,+1}\mathcal{Q}_1 + \delta_{\eta,-1}\mathcal{Q}_2 x}{x^2-1} + \eta\left(-H_1 - \bar{H}_3 + \bar{H}_4\right) + \phi_1 \\
p_2 &= +\frac{2\pi\mathcal{J}x - \delta_{\eta,-1}\mathcal{Q}_1 + \delta_{\eta,+1}\mathcal{Q}_2 x}{x^2-1} + \eta\left(-H_1 + H_2 + \bar{H}_2 - \bar{H}_3\right) + \phi_2 \\
p_3 &= +\frac{2\pi\mathcal{J}x - \delta_{\eta,-1}\mathcal{Q}_1 + \delta_{\eta,+1}\mathcal{Q}_2 x}{x^2-1} + \eta\left(-H_2 + H_3 + \bar{H}_1 - \bar{H}_2\right) + \phi_3 \\
p_4 &= +\frac{2\pi\mathcal{J}x - \delta_{\eta,+1}\mathcal{Q}_1 + \delta_{\eta,-1}\mathcal{Q}_2 x}{x^2-1} + \eta\left(+H_3 - H_4 + \bar{H}_1\right) + \phi_4 \\
p_5 &= -\frac{2\pi\mathcal{J}x - \delta_{\eta,+1}\mathcal{Q}_1 + \delta_{\eta,-1}\mathcal{Q}_2 x}{x^2-1} + \eta\left(-H_5 + H_4 - \bar{H}_7\right) + \phi_5 \\
p_6 &= -\frac{2\pi\mathcal{J}x - \delta_{\eta,-1}\mathcal{Q}_1 + \delta_{\eta,+1}\mathcal{Q}_2 x}{x^2-1} + \eta\left(-H_5 + H_6 + \bar{H}_6 - \bar{H}_7\right) + \phi_6 \\
p_7 &= -\frac{2\pi\mathcal{J}x - \delta_{\eta,-1}\mathcal{Q}_1 + \delta_{\eta,+1}\mathcal{Q}_2 x}{x^2-1} + \eta\left(-H_6 + H_7 + \bar{H}_5 - \bar{H}_6\right) + \phi_7 \\
p_8 &= -\frac{2\pi\mathcal{J}x - \delta_{\eta,+1}\mathcal{Q}_1 + \delta_{\eta,-1}\mathcal{Q}_2 x}{x^2-1} + \eta\left(+H_7 + \bar{H}_5 - \bar{H}_4\right) + \phi_8
\end{aligned} \tag{6.4}$$

where $G_4(x) \equiv -\sum_{n=0}^{\infty} \mathcal{Q}_{n+1}x^n$. We can also write

$$\frac{2\pi}{\sqrt{\lambda}}\,\delta\mathcal{D} = \mathcal{Q}_2\,.$$

Then, to leading order, these quasi-momenta define an eight-sheet Riemann surface and the BS equations read simply $\not{p}_i - \not{p}_j = 2\pi n_{ij}$ in each of the cuts $\mathcal{C}_{ij}$ uniting p_i and p_j. Finally, in this section we will use

$$\cot_{\rm ij} \equiv \alpha(x)\frac{p'_{\rm i} - p'_{\rm j}}{2}\cot\frac{p_{\rm i} - p_{\rm j}}{2}$$

which is similar (but should not be confused) with (3.3).

[13]note that

$$F_a(x) = G_a(x) + \bar{G}_a(x) = H_a(x) + \bar{H}_a(x)\,.$$

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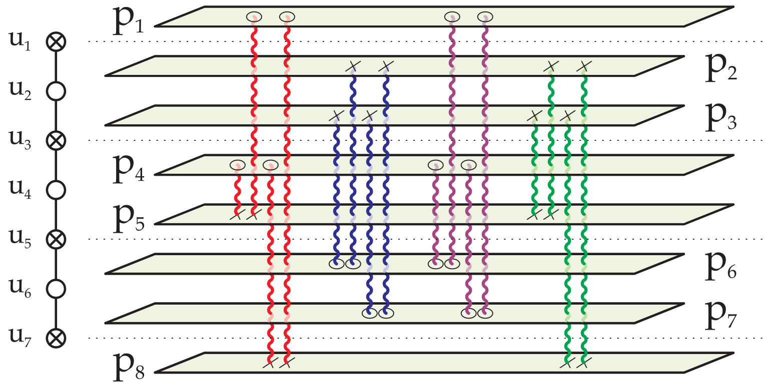

Figure 11: The several physical fluctuations in the string Bethe ansatz. The 16 elementary physical excitations are the stacks (bound states) containing the middle node root. From the left to the right we have four S^5 fluctuations, four AdS_5 modes and eight fermionic excitations. The bosonic (fermionic) stacks contain an even (odd) number of fermionic roots represented by a cross in the $psu(2,2|4)$ Dynkin diagram in the left.

6.2 Middle node anomaly

In this section we will expand BS equations in the scaling limit for the roots belonging to a cut containing middle node roots x_4 only. We do not assume that all the others cuts are of the same type, rather they can be cuts of stacks of several sizes. In the section 5.3 we will generalize the results obtained in this section to an arbitrary cut, assuming, as in the previous section, that the cuts are small enough and twists are not zero so that stacks are stable. We will discus in section 6.7 what happens when one takes all twists to zero.

To leading order, the middle node equation (6.1) can be simply written as $\not{p}_4 - \not{p}_5 = 2\pi n$ while at 1-loop the first product in the r.h.s. of (6.1) corrects this equation due to

$$\frac{1}{i}\log\prod_{j\neq k}^{K_4}\left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\right) \simeq 2\,\not{F}_4(x) + \alpha(x)\pi\rho'(x)\cot(\pi\rho(x)) \tag{6.5}$$

where $\rho(x) = \frac{dk}{du_k}$. Expansion of the remaining terms in (6.1) will not lead to the appearance of such *anomaly* like terms since the roots of another types are separated by ~ 1 from $x_{4,k}$. Thus we have simply

$$2\pi n = \not{p}_4 - \not{p}_5 - \eta\,\alpha(x)\pi\rho'(x)\cot(\pi\rho(x)) \quad , \quad x\in\mathcal{C}_{45}$$

In the next sections we will use dualities of the BS equations to get some extra information about cuts of stacks and generalize the above equation to any possible type of cut. To achieve this we shall recast this equation in terms of the middle node roots x_4 only.

6.3 Dualities in the string Bethe ansatz

Obviously, the behavior of the Bethe roots will be as described in section 2 for a simpler example of a $su(1,2)$ spin chain, that is, we will have simple cuts made out of x_4 roots only and also cuts of stacks with x_2, x_3 and x_4 roots for example. Consider such cut of stacks.

Clearly, to be able to write the middle node equation (6.1) or (6.6) we need to compute the density mismatches $\rho_2 - \rho_3$ and $\rho_3 - \rho_4$ which are 1-loop contributions we must take into account if we want to write an integral equation for the middle node equation in terms of the density ρ_4 of momentum carrying roots only. In this section we shall analyze the dualities present in the BS Bethe equations. By analyzing them in the scaling limit we will then be able to derive the desired density mismatches.

6.3.1 Fermionic duality in scaling limit

In [25] it was shown that the BS equations obey a very important fermionic duality. Since we chose to work with a subset of the possible Bethe equations, that is the ones with $\eta_1 = \eta_2 = \eta$ present in [25], we should apply the duality present below not only to the fermionic roots x_1 and x_3 (as described below) but also to the Bethe roots x_5 and x_7. Obviously the duality for x_5 and x_7 is exactly the same as for x_1 and x_3 and so we will focus simply on the latter while keeping implicit that we always dualize all the fermionic roots at the same time.

We construct the polynomial ($\tau = \eta\,(\phi_4 - \phi_3)$)

$$\begin{aligned} P(x) = \; & e^{+i\frac{\tau}{2}} \prod_{j=1}^{K_4} (x - x_{4,j}^+) \prod_{j=1}^{K_2} (x - x_{2,j}^-)(x - 1/x_{2,j}^-) \\ & - e^{-i\frac{\tau}{2}} \prod_{j=1}^{K_4} (x - x_{4,j}^-) \prod_{j=1}^{K_2} (x - x_{2,j}^+)(x - 1/x_{2,j}^+) \end{aligned} \tag{6.6}$$

of degree $K_4 + 2K_2$ which clearly admits $x = x_{3,j}$ and$x = 1/x_{1,j}$ as $K_3 + K_1$ zeros.[14] The remaining $K_4 + 2K_2 - K_3 - K_1$ roots are denoted by $\tilde{x}_{3,j}$ or $1/\tilde{x}_{1,j}$ depending on whether they are outside or inside the unit circle respectively,

$$P(x) = 2i\sin(\tau/2) \prod_{j=1}^{K_1} (x - 1/x_{1,j}) \prod_{j=1}^{\tilde{K}_1} (x - 1/\tilde{x}_{1,j}) \prod_{j=1}^{K_3} (x - x_{3,j}) \prod_{j=1}^{\tilde{K}_3} (x - \tilde{x}_{3,j}) \tag{6.7}$$

Then we can replace the roots $x_{1,j}, x_{3,j}$ by the roots $\tilde{x}_{1,j}, \tilde{x}_{3,j}$ in the BS equations provided we change the grading $\eta \to -\eta$ and interchange the twists $\phi_1 \leftrightarrow \phi_2$ and $\phi_3 \leftrightarrow \phi_4$. In fact, since we should also dualize the remaining fermionic roots, we should also change $\phi_5 \leftrightarrow \phi_6$ and $\phi_7 \leftrightarrow \phi_8$ and replace the remaining fermionic roots x_5 and x_7.

Since to the leading order $x^\pm \simeq x$ each root will belong to a stack which must always contain a momentum carrying root x_4. We have therefore $\tilde{K}_1 = K_2 - K_1$ and $\tilde{K}_3 = K_2 + K_4 - K_3$. Thus we label the Bethe roots as

$$\begin{aligned} x_{1,j} &= x_{4,j} - \epsilon_{1,j} \;, && j = 1, \ldots, K_1 \\ \tilde{x}_{1,j} &= x_{4,j+K_1} - \tilde{\epsilon}_{1,j} \;, && j = 1, \ldots, \tilde{K}_1 \\ x_{2,j} &= x_{4,j} - \epsilon_{2,j} \;, && j = 1, \ldots, K_2 \end{aligned}$$

[14] we also have $1/x_1$ has zeros because, due to (6.2), the equation for $x_{1,j}$ is the same as the equation for $x_{3,j}$ if we replace $x_{3,j}$ by $1/x_{1,j}$. This is why the restriction (6.2) of the twists is so important.

$$x_{3,j} = x_{4,j} - \epsilon_{3,j} \ , \qquad\qquad j = 1, \dots, K_3$$
$$\tilde{x}_{3,j} = x_{4,j+K_3} - \tilde{\epsilon}_{3,j} \ , \qquad\qquad j = 1, \dots, \tilde{K}_3$$

with $\epsilon \sim 1/\sqrt{\lambda}$. Dividing (6.6) and (6.7) by $\prod_{j=1}^{K_4}(x - x_{4,j})\prod_{j=1}^{K_2}(x - x_{4,j})(x - 1/x_{4,j})$ we have

$$e^{+i\frac{\tau}{2}} \prod_{j=1}^{K_4} \frac{x - x_{4,j}^+}{x - x_{4,j}} \prod_{j=1}^{K_2} \frac{x - x_{2,j}^-}{x - x_{4,j}} \frac{x - 1/x_{2,j}^-}{x - 1/x_{4,j}} - e^{-i\frac{\tau}{2}} \prod_{j=1}^{K_4} \frac{x - x_{4,j}^+}{x - x_{4,j}} \prod_{j=1}^{K_2} \frac{x - x_{2,j}^+}{x - x_{4,j}} \frac{x - 1/x_{2,j}^+}{x - 1/x_{4,j}} \tag{6.8}$$
$$= 2i \sin(\tau/2) \prod_{j=1}^{K_1} \frac{x - 1/x_{1,j}}{x - 1/x_{4,j}} \prod_{j=1}^{\tilde{K}_1} \frac{x - 1/\tilde{x}_{1,j}}{x - 1/x_{4,K_1+j}} \prod_{j=1}^{K_3} \frac{x - x_{3,j}}{x - x_{4,j}} \prod_{j=1}^{\tilde{K}_3} \frac{x - \tilde{x}_{3,j}}{x - x_{4,K_3+j}}$$

In this form it is easy to expand the duality relation in powers of $1/\sqrt{\lambda}$. By expanding all factors in (6.8) such as

$$\prod_{j=1}^{K_2} \frac{x - x_{2,j}^\pm}{x - x_{4,j}} = \exp\left(\sum_{j=1}^{K_2} \log \frac{x - x_{2,j}^\pm}{x - x_{4,j}}\right) \simeq \exp\left(\mp \frac{i}{2} G_2(x) + \sum_j^{K_2} \frac{\epsilon_{2,j}}{x - x_{2,j}}\right) \ ,$$

we find

$$\sin\left(\frac{\eta(p_4 - p_3)}{2}\right) = \sin\left(\frac{\tau}{2}\right) \exp\left(+\sum \frac{\epsilon_3}{x - x_3} + \sum \frac{\tilde{\epsilon}_3}{x - x_3} - \sum \frac{\epsilon_2}{x - x_2}\right)$$
$$\times \exp\left(-\sum \frac{\epsilon_1/x_1^2}{x - 1/x_1} - \sum \frac{\tilde{\epsilon}_1/\tilde{x}_1^2}{x - 1/\tilde{x}_1} + \sum \frac{\epsilon_2/x_2^2}{x - 1/x_2}\right) .$$

Then, similarly to what we had in section 3.2 for the bosonic duality, we notice that

$$\alpha(x)\partial_x \left(\sum \frac{\epsilon_3}{x - x_3} + \sum \frac{\tilde{\epsilon}_3}{x - \tilde{x}_3} - \sum \frac{\epsilon_2}{x - \tilde{x}_2}\right) = H_3 + H_{\tilde{3}} - H_4 - H_2 \ ,$$

with a similar expression for the argument of the second exponential. Thus finally we get

$$(H_4 + H_2 - H_3 - H_{\tilde{3}}) + \left(\bar{H}_2 - \bar{H}_1 - \bar{H}_{\tilde{1}}\right) = -\cot_{34} \ ,$$

or alternatively, using the $x \to 1/x$ symmetry transformation properties of the quasi-momenta,

$$\left(\bar{H}_4 + \bar{H}_2 - \bar{H}_3 - \bar{H}_{\tilde{3}}\right) + (H_2 - H_1 - H_{\tilde{1}}) = -\cot_{12} \ .$$

From this expressions we can deduce several properties of the density mismatches we wanted to obtain. For example, if we compute the discontinuity of (6.3.1) at a cut containing roots x_1, that is in a large cut of stacks $\mathcal{C}_{1,i>4}$, we immediately get

$$\rho_1 - \rho_2 = -\frac{\Delta \cot_{12}}{2\pi i} \ , \quad x \in \mathcal{C}_{1,i>4} \, . \tag{6.9}$$

Proceeding in a similar way we find

$$\rho_3 - \rho_4 = -\frac{\Delta \cot_{34}}{2\pi i} \ , \qquad\qquad x \in \mathcal{C}_{3,i>4} \, , \tag{6.10}$$
$$\rho_3 - \rho_4 = \rho_2 - \rho_{\tilde{3}} \ , \qquad\qquad x \in \mathcal{C}_{1,i>4} \cup \mathcal{C}_{2,i>4} \, . \tag{6.11}$$

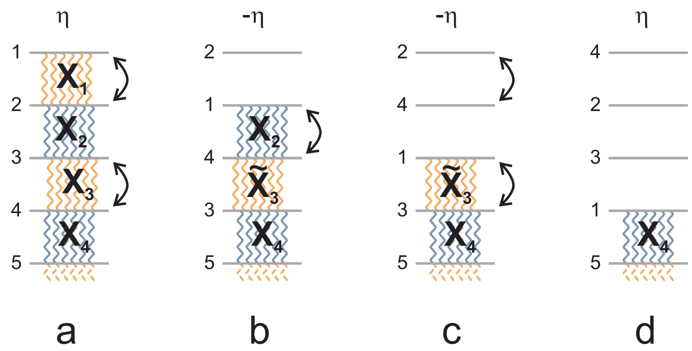

Figure 12: Action of the duality on a long stack. By successively applying the fermionic and the bosonic dualities duality we can reduce the size of any large cut. One should not forget to change the sign of the grading η after applying the fermionic duality.

Let us now show that in the scaling limit the fermionic duality corresponds just to the exchange of the sheets $\{p_i\}$ of the Riemann surface. For illustration let us pick p_1 and see how it transforms under the duality. By definition the fermionic duality corresponds to the replacement $\eta \to -\eta, H_1 \to H_{\tilde{1}}, H_3 \to H_{\tilde{3}}$ and $\phi_1 \leftrightarrow \phi_2$, $\phi_3 \leftrightarrow \phi_4$, so that

$$p_1 \to \frac{2\pi\mathcal{J}x - \delta_{\eta,-1}Q_1 + \delta_{\eta,+1}Q_2 x}{x^2-1} - \eta\left(-H_{\tilde{1}} - \bar{H}_{\tilde{3}} + \bar{H}_4\right) + \phi_2 = p_2 + \eta\cot_{12}$$

In the same way we get

$$p_2 \to p_1 + \eta\cot_{12} \quad , \quad p_3 \to p_4 - \eta\cot_{34} \quad , \quad p_4 \to p_3 - \eta\cot_{34} \ ,$$

and since $\cot_{ij} \sim 1/\sqrt{\lambda}$ we see that to the leading order the duality indeed just exchanges the sheets.

6.3.2 Bosonic duality in scaling limit

The bosonic nodes of the BS equations are precisely as in the usual Bethe ansatz discussed in the first sections so that we can just briefly mention the results. The duality ($\tau = \eta(\phi_2 - \phi_3)$)

$$e^{+i\frac{\tau}{2}}\tilde{Q}_2(u-i/2)Q_2(u+i/2) - e^{-i\frac{\tau}{2}}\tilde{Q}_2(u+i/2)Q_2(u-i/2) = 2i\sin\frac{\tau}{2}Q_1(u)Q_3(u)$$

leads to

$$(H_1 + H_3 - H_2 - H_{\tilde{2}}) + (\bar{H}_1 + \bar{H}_3 - \bar{H}_2 - \bar{H}_{\tilde{2}}) = \cot_{23} \tag{6.12}$$

which implies

$$\rho_2 - \rho_3 = +\frac{\Delta\cot_{23}}{2\pi i} \ , \ x \in \mathcal{C}_{2,i>4}$$

As we already discussed in section 2 the bosonic duality also amounts to an exchange of Riemann sheets. Indeed, under the replacement $H_2 \to H_{\tilde{2}}$ and $\phi_2 \leftrightarrow \phi_3$, we find

$$p_2 \to p_3 - \eta\cot_{23} \quad , \quad p_3 \to p_2 + \eta\cot_{23}$$

which again, to the leading order in $\sqrt{\lambda}$, is just the exchange of the sheets of the curve.

	$\mathcal{C}_{1,i}$	$\mathcal{C}_{2,i}$	$\mathcal{C}_{3,i}$
$2\pi i(\rho_1-\rho_2)$	$-\Delta\cot_{12}$		
$2\pi i(\rho_2-\rho_3)$	$-\Delta\cot_{13}$	$+\Delta\cot_{23}$	
$2\pi i(\rho_3-\rho_4)$	$+\Delta\cot_{14}$	$-\Delta\cot_{24}$	$-\Delta\cot_{34}$

Table 1: Densities missmatches.

6.3.3 Dualities and the missing mismatches

Using bosonic and fermionic dualities separately we already got some information about the several possible mismatches of the densities inside the stack. To compute the missing mismatches we have to use both dualities together. For example suppose we want to compute $\rho_3-\rho_4$ in a cut $\mathcal{C}_{1,i>4}$. We start by one such large cut of stacks (see figure 12a) and we apply the fermionic duality to this configuration so that we obtain a smaller cut as depicted in figure 12b. For this configuration we can use (6.3.2) to get

$$\rho_2-\rho_{\tilde{3}}=+\frac{\Delta\cot_{14}}{2\pi i}\,.$$

However, from (6.11), this is also equal to the mismatch we wanted to compute, that is

$$\rho_3-\rho_4=+\frac{\Delta\cot_{14}}{2\pi i}\quad,\quad x\in\mathcal{C}_{1,i>4}\,.$$

To compute the last mismatch we apply the bosonic duality to get a yet smaller cut as in figure 12c for which we use (6.10) to get

$$\rho_{\tilde{3}}-\rho_4=-\frac{\Delta\cot_{13}}{2\pi i}\,.$$

Again, from (6.11), we can revert this result into a mismatch for the configuration before duality, that is

$$\rho_2-\rho_3=-\frac{\Delta\cot_{13}}{2\pi i}\quad,\quad x\in\mathcal{C}_{1,i>4}\,.$$

Let us then summarize all densities mismatches in table 1.

6.4 Integral equation

In this section we shall recast equation (6.6) or

$$\eta\frac{4\pi\mathcal{J}x-2\delta_{\eta,+1}\mathcal{Q}_1-2\delta_{\eta,-1}\mathcal{Q}_2x}{x^2-1}+2\not{H}_4-H_3-H_5-\bar{H}_1-\bar{H}_7=2\pi n+\eta\phi_4-\eta\phi_5-\cot_{45}\quad(6.13)$$

in terms of the density $\rho_4(x)$ of the middle roots x_4. To do so we only need to replace the several densities by the middle node density $\rho_4(x)$ using the several density mismatches presented in table 1. Defining

$$H_{ij}(x)\equiv\int\limits_{\mathcal{C}_{ij}}\frac{\alpha(x)}{\alpha(y)}\frac{\rho_4(y)}{x-y}dy$$

we can then rewrite equation (6.13) in terms of the middle node roots only,

$$\eta\frac{4\pi\mathcal{J}x-2\delta_{\eta,+1}\mathcal{Q}_1-2\delta_{\eta,-1}\mathcal{Q}_2x}{x^2-1}+2\,\not{H}_{45}+H_{15}+H_{48}-2\bar{H}_{18}-\bar{H}_{15}-\bar{H}_{48} \tag{6.14}$$
$$=2\pi n+\eta\phi_4-\eta\phi_5-\cot_{45}+\sum_{\substack{1\le i\le 4\\5\le j\le 8}}\left(\mathcal{I}^{i4}_{ij}+\mathcal{I}^{5j}_{ij}\right)+\sum_{\substack{1\le i\le 4\\5\le j\le 8}}\left(\bar{\mathcal{I}}^{i1}_{1j}+\bar{\mathcal{I}}^{8j}_{i8}\right)$$

where $x\in\mathcal{C}_{45}$ and

$$\mathcal{I}^{kl}_{ij}(x)=(-1)^{F_{kl}}\int\limits_{\mathcal{C}_{ij}}\frac{\alpha(x)}{\alpha(y)}\frac{\Delta\cot_{kl}}{x-y}\frac{dy}{2\pi i}\ \ ,\ \ \mathcal{I}^{kk}_{ij}(x)\equiv 0\ \ ,\ \ \bar{\mathcal{I}}^{kl}_{ij}(x)=\mathcal{I}^{kl}_{ij}(1/x)\,.$$

The several dualities amount to an exchange of Riemann sheets so that the cuts $\mathcal{C}_{ij}\to\mathcal{C}_{i'j'}$ with the subscripts in H_{ij} changing accordingly. The middle roots x_4 are never touched in the process. Moreover to leading order $p_i\leftrightarrow p_{i'}$ and thus the r.h.s. of (6.14) is also trivially changed under the dualities. Therefore, as in section 3 (see (3.4) and (3.5)), we can now trivially write the corrected equation when x belongs to any possible type of cut of stacks by applying the several dualities to equation (6.14).

6.5 Fluctuations

In this section we shall find the integral equation (6.14) from the field theoretical point of view like we did in section 4.1 and in appendix B. That is, we will find what the corrections to the classical (leading order) equations [9]

$$\eta\frac{4\pi\mathcal{J}x-2\delta_{\eta,+1}\mathcal{Q}_1-2\delta_{\eta,-1}\mathcal{Q}_2x}{x^2-1}+2\not{H}_4-H_3-H_5-\bar{H}_1-\bar{H}_7=2\pi n+\eta\phi_4-\eta\phi_5\,, \tag{6.15}$$

should be in order to describe properly the semi-classical quantization of the string (and not only the classical limit). We will find that this construction leads precisely to the integral equation (6.14) thus showing that the BS nested Bethe ansatz equations do reproduce the 1-loop shift around any (stable) classical solution with exponential precision (in some large charge of the classical solution). This section is very similar to section 4 and to appendix B and thus we will often omit lengthy but straightforward intermediate steps. We assume $i=1,\ldots,4$ and $j=5,\ldots,8$ in all sums.

As in (4.6) and (B.1), we add $\frac{1}{2}(-1)^F$ of a virtual excitation for each possible mode number n and polarization ij to each quasi-momenta. Notice that for this super-symmetric model the fluctuations can also be fermionic and indeed the grading $(-1)^F$ equals $+1$ (-1) for bosonic (fermionic) fluctuations, see figure 11, as usual for bosonic (fermionic) harmonic oscillators.

We denote $\rho=\rho_0+\delta\rho$ where ρ_0 is the leading density, solution of the leading (classical) equation (6.15), while ρ obeys the corrected (semi-classical) equation. For example, if we consider $x \in \mathcal{C}_{4,5}$, the starting point should be (see [32] for a similar analysis)

$$\begin{aligned}
&\frac{-2x\delta_{\eta,-1}\delta\mathcal{Q}_1}{x^2-1} + 2\int\limits_{\mathcal{C}_{45}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y} + \int\limits_{\mathcal{C}_{15}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y} + \int\limits_{\mathcal{C}_{48}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y} \\
&\quad -2\int\limits_{\mathcal{C}_{18}} \frac{\alpha(1/x)}{\alpha(y)}\frac{\delta\rho(y)}{1/x-y} - \int\limits_{\mathcal{C}_{15}} \frac{\alpha(1/x)}{\alpha(y)}\frac{\delta\rho(y)}{1/x-y} - \int\limits_{\mathcal{C}_{48}} \frac{\alpha(1/x)}{\alpha(y)}\frac{\delta\rho(y)}{1/x-y} \\
&\quad + \sum_{n=-N}^{N} \frac{1}{2}\left[\sum_{i<4} \frac{\alpha(x)}{x-x_n^{i5}} + \sum_{j>5} \frac{\alpha(x)}{x-x_n^{4j}} - \sum_{i<4} \frac{\alpha(1/x)}{1/x-x_n^{i8}} - \sum_{j>5} \frac{\alpha(1/x)}{1/x-x_n^{1j}}\right] = 0
\end{aligned} \tag{6.16}$$

Then, by construction, the charges

$$Q_r = \int_{\mathcal{C}} \frac{\rho(y)}{y^r}dy + \sum_n \sum_{ij} (-1)^{F_{ij}} \frac{\alpha(x_n^{ij})}{2(x_n^{ij})^r} = \int_{\mathcal{C}} \frac{\rho(y)}{y^r}dy + \sum_{ij} \frac{(-1)^{F_{ij}}}{2} \oint_{x_n^{ij}} \frac{\cot_{ij}}{y^r}\frac{dy}{2\pi i} \tag{6.17}$$

will take the $1/\sqrt{\lambda}$ corrected values. It is clear that, as before, we do not include the new *virtual* excitations in the density $\rho(x)$. Similarly to (4.12) and (B.6), if we want the charges to have the standard form

$$\mathcal{Q}_r = \int \frac{\varrho(y)}{y^r}dy$$

we must redefine the density as

$$\varrho = \rho + \frac{1}{4\pi i}\left(\sum_{i<i'\leq 4} (-1)^{F_{ii'}}\Delta\cot_{ii'} + \sum_{j>j'\geq 5} (-1)^{F_{jj'}}\Delta\cot_{jj'}\right).$$

Now we want to go back to the integral equation (6.16) and rewrite it using the density $\delta\varrho = \varrho - \rho_0$. For example, for $x \in \mathcal{C}_{45}$,

$$\begin{aligned}
&2\int\limits_{\mathcal{C}_{45}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y} + \int\limits_{\mathcal{C}_{15}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y} + \int\limits_{\mathcal{C}_{48}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y} \\
&+ \sum_{n=-N}^{N} \frac{1}{2}\left[\sum_i \frac{(-1)^{F_{i5}}\alpha(x)}{x-x_n^{i5}} + \sum_j \frac{(-1)^{F_{4j}}\alpha(x)}{x-x_n^{4j}}\right] = \\
&\qquad 2\int\limits_{\mathcal{C}_{45}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\varrho(y)}{x-y} + \int\limits_{\mathcal{C}_{15}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\varrho(y)}{x-y} + \int\limits_{\mathcal{C}_{48}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\varrho(y)}{x-y} \\
&\qquad + \cot_{45} - \sum_{ij}\left(\mathcal{I}_{ij}^{4i} + \mathcal{I}_{ij}^{j5}\right) - \frac{1}{2}\sum_{ij}\left(\bar{\mathcal{I}}_{8j}^{8i} + \bar{\mathcal{I}}_{1i}^{1j} + \bar{\mathcal{I}}_{ij}^{8i} + \bar{\mathcal{I}}_{ij}^{1j}\right)
\end{aligned}$$

where the identity

$$(-1)^{F_{4i}}\cot_{4,i} = -\sum_j \left(\mathcal{I}_{4j}^{4i} + \mathcal{I}_{ij}^{4i}\right) - \sum_j \left(\bar{\mathcal{I}}_{1j}^{1\bar{i}} + \bar{\mathcal{I}}_{\bar{i}j}^{1\bar{i}}\right),$$

where $\bar{\bar{i}} = i,\ \bar{1} = 4,\ \bar{2} = 3$, is being used. Now, when $x \in \mathcal{C}_{18}$, we will get

$$2\int\limits_{\mathcal{C}_{18}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y} + \int\limits_{\mathcal{C}_{15}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y} + \int\limits_{\mathcal{C}_{48}} \frac{\alpha(x)}{\alpha(y)}\frac{\delta\rho(y)}{x-y}$$

$$+\sum_{n=-N}^{N}\frac{1}{2}\left[\sum_i \frac{(-1)^{F_{i8}}\alpha(x)}{x-x_n^{i8}}+\sum_j \frac{(-1)^{F_{1j}}\alpha(x)}{x-x_n^{1j}}\right]=$$

$$2\int_{\mathcal{C}_{18}}\frac{\alpha(x)}{\alpha(y)}\frac{\delta\varrho(y)}{x-y}+\int_{\mathcal{C}_{15}}\frac{\alpha(x)}{\alpha(y)}\frac{\delta\varrho(y)}{x-y}+\int_{\mathcal{C}_{48}}\frac{\alpha(x)}{\alpha(y)}\frac{\delta\varrho(y)}{x-y}$$
$$-\frac{1}{2}\sum_{ij}\left(\mathcal{I}_{ij}^{1i}+\mathcal{I}_{ij}^{j8}-\mathcal{I}_{8j}^{8i}-\mathcal{I}_{1i}^{1j}\right)$$

Finally we can use the x to $1/x$ symmetry to translate last equality into one for $x\in\mathcal{C}_{45}$. Subtracting it from the previous equation we see that the $1/\sqrt{\lambda}$ corrected equation will correspond to adding

$$-\cot_{45}+\sum_{ij}(\mathcal{I}_{ij}^{4i}+\mathcal{I}_{ij}^{5i}+\bar{\mathcal{I}}_{1j}^{1i}+\bar{\mathcal{I}}_{8i}^{8j})$$

to the r.h.s. of (6.15) thus obtaining, after the identification $\varrho=\rho_4$, precisely the finite size corrected equation (6.14) obtained from the NBA point of view!

6.6 The unit circle and the Hernandez-Lopez phase

In the last section we showed that the one loop shift as a sum of all fluctuation energies (or others local charges) perfectly matches the finite size corrections in the NBA equations. However we systematically dropped the contours around the unit circle.

For example, when we blow the contour in the last term of (6.17), we also get some contribution from the singularities inside the unit circle. That is we will have an extra contribution to the charges given by an integral over the unit circle. Also, take (6.16) for instance. To pass to the r.h.s we transformed the collections of poles into integrals over the excitation points and then we blew the contour which became a collection of contours on the several existing cuts. Again we dropped the contribution from the integrals over the unit circle which would lead to an extra $1/\sqrt{\lambda}$ term in the r.h.s. of (6.14). In our previous paper [32] we showed[15] that this extra contribution matches precisely the extra contribution coming from the Hernandez-Lopez phase in the NBA!

However, as we explained in [32], in order to obtain precisely the HL phase a precise prescription for the labeling of the mode numbers of the fluctuations must be given.

Moreover, in [32], we assumed that everywhere we can replace $\cot\left(\frac{p_i(x)-p_j(x)}{2}\right)$ by $i\,\mathrm{sign}(\mathrm{Im}\,x)$ with exponential precision in $\frac{L}{\sqrt{\lambda}}$. This is reasonable for generic points in the unit circle, where the imaginary part of $p_i(x)-p_j(x)$ is large, but one has to carefully analyze the neighbourhood of the real axis, where this imaginary part vanishes.

Let us consider these two subtle points in greater detail.

6.6.1 A mode number prescription

As we emphasized in [31] if we number the fluctuation charges $\mathcal{Q}_n^{ij}$ differently we might obtain different results for the 1-loop shift, that is for the graded sums of these fluctuation

[15] Recently the HL phase was also found [78] in the study of the open string scattering of giant magnons [79].

charges. Thus a precise prescription for the labeling of the quantum fluctuations is crucial. In the appendix A of [32] we found out that the contribution of the integrals of the previous section does reproduce the HL phase provided we number the quantum fluctuations located at x_n^{ij} according to

$$p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi\,(n - m_i + m_j)$$

with some specific choice of m_i. Moreover we also showed that for the same choice of m_i the contribution to the charges coming from the above mentioned integrals over the unit circle is zero. Using the x to $1/x$ symmetries following from the definition of the quasi-momenta (6.4) plus the restriction (6.2) on the twists, we can redo the computation in the appendix A of [32] to find that the condition on the m_i now reads

$$(m_2 + m_3 - m_1 - m_4)\,(m_5 + m_8 - m_6 - m_7) = 0$$

so that, in particular, $m_i = 0$ does the job nicely. We see that, with the introduction of these twists and subsequent redefinition of the quasimomenta, the prescription for the labeling of the excitations becomes absolutely natural and algebraic curve friendly [31]. This answers the question raised in [32] concerning the naturalness of the presciption needed to obtain the HL phase [39] – see appendix A in [32].

6.6.2 Unit circle contribution

Let us now us focus on the vicinity of $x = 1$ where we have the following expansion of the quasi-momenta

$$\frac{p_i(x) - p_j(x)}{2} = \frac{\beta_{ij}}{x-1} + \dots$$

where β_{ij} is usually of order $L/\sqrt{\lambda}$ (and should be so for the asymptotical BAE to be valid). We will consider the circle with radius $x^{ij}_{N+1/2} \simeq 1 + \frac{1}{\pi N \beta_{ij}}$, where N is some large cutoff in the sum of fluctuations (6.16). We want to estimate

$$\int \alpha(x) f(x) \left[\cot\left(\frac{p_i - p_j}{2}\right) + i\,\text{sign}(\text{Im}\,x)\right](p_i' - p_j')dx\,.$$

This integral is dominated for $x \simeq \pm 1$ and can be performed by saddle point. The contribution for $x \simeq 1$ is

$$\int \alpha(x) f(x) \left[\cot\left(\frac{p_i - p_j}{2}\right) + i\,\text{sign}(\text{Im}\,x)\right](p_i' - p_j')dx = \frac{i\pi^3 f(1)}{6\beta_{ij}\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{N}\right)$$

which is zero under the sum over all polarizations. For example

$$\frac{(-1)^{F_{45}}}{\beta_{45}} = -\frac{(-1)^{F_{35}}}{\beta_{35}}\,.$$

Thus we can indeed drop the cot's when integrating over the unit circle and thus we finally conclude that the one loop shift to any local charge computed from the BS equations with the Hernandez-Lopez phase is indeed given by the sum of fluctuations as predicted by field theoretical arguments.

6.7 Zero twist and large fillings via analytical continuation

Although we always assumed the twists to be sufficiently large and the fillings to be sufficiently small we can always analytically continue the results towards zero twists or large filling fractions. Let us briefly explain why. In the scaling limit, for large twists, the bosonic duality we introduced amounts to a simple exchange of sheets in some Riemann surface, $p_a(x) \leftrightarrow p_b(x)$. As we saw in section 5.3 (see also section 5.4) what happens when the twists start to become very small is that the quasi-momenta are still simply exchanged but in a piecewise manner, that is, we can always split the complex planes in some finite number of regions where the bosonic duality simply means $p_a(x) \leftrightarrow p_b(x)$. Thus, from the e^{ip} algebraic curve point of view nothing special occurs for what analyticity is concerned and therefore we can safely analytically continue our findings to any value of the twists. Exactly the same analysis holds for the filling fractions. Moreover, for the usual Bethe system, we defined a set of quasi-momenta, which constitute an algebraic curve to any order in $1/L$, and therefore we don't expect analyticity to break down at any order in $1/L$.

We also preformed a high precision numerical check concluding that there is no singularity when the configuration of the Bethe roots is affected by this partial reshuffling of the sheets and that finite size corrections are still related to the same sum of fluctuations, which are analytical functions w.r.t. the twists. To be more specific, we can compute the correction a single cut solution in the $su(2)$ setup with or without the presence of the condensates described in section 5.3.2. The presence or absence of such condensates for a single cut solution can be controlled by either playing with the values of the twists or by changing the filling fraction of the cut. We can see that as this manipulations are preformed no mismatch with the expected 1-loop shift is generated. See [54] for more details on such manipulations.

7. Conclusions

In this paper we studied generic nested Bethe ansatz (NBA) equations, the corresponding scaling limit and its leading finite size corrections. Let us summarize briefly our main results

- We found out that the introduction of some extra phases, called twists, are crucial for the formation of bound states of roots of different types, called in the literature by stacks [18]. Strictly speaking without these twists the stacks do not exist. See sections 2 and 5.3.
- We understood how to use the bosonic duality between various systems of Bethe roots which is present even in the absence of any fermionic symmetry. In the scaling limit we showed that this duality amounts to a reshuffling of Riemann sheets of the algebraic curve formed by the condensation of Bethe roots. See sections 2 and 5.
- We explained how to write down the integral equation describing the leading finite size corrections around generic NBA's for (super) spin chains by using the transfer matrices for (super) group along with some TQ relations. See section 3.1
- We provided an alternative derivation of this integral equation using an independent path, namely using the dualities present in the Bethe equations allowing one to get

rid of the several stacks and reduce the size of any cut by successive application of several dualities. See section 3.2.

- We obtained the integral equation describing the finite size corrections to the Beisert-Staudacher equations [25] with the Hernandez-Lopez phase [11, 39] in the scaling limit (to do so we were forced to use the duality approach because at present the $psu(2,2|4)$ transfer matrices for this Bethe ansatz are not known[16]). See section 6.
- In the scaling limit Beisert-Staudacher equations [25] describe the classical motion of the superstring on $AdS_5 \times S^5$ through the finite gap curves of [9]. Thus the integral equation we found should reproduce the 1-loop shift for all the charges around any classical string motion and this is obviously a very nontrivial check of the validity of the BS equations. We show that this equation indeed mimics the presence of a sea of virtual particles thus proving this general statement. See section 6.5.

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A. Transfer matrix invariance and the bosonic duality for SU($K|M$) supergroups

In this section we review the formalism of [24] which allows one to derive the transfer matrices of usual (super) spin chains in any representation. We will use this general formalism to prove the invariance under the bosonic dualities of all possible transfer matrices one can build. The transfer matrices presented in section 3.1 can be obtained trivially using this formalism.[17]

As mentioned in section 2, for the standard SU($K|M$) super spin chains (based on the standard R-matrix $R(u) = u + i\mathcal{P}$ with $\mathcal{P}$ the super permutation) we can find the (twisted) transfer matrix eigenvalues for the single column young tableau with a boxes through the *non-commutative generating functions* [24, 40]

$$\sum_{a=0}^{\infty}(-1)^a e^{ia\partial_u}\frac{T_a(u)}{Q_{K,M}(u+(a-K+M+1)\,i/2)}e^{ia\partial_u} = \overrightarrow{\prod}_{(x,n)\in\gamma}\hat{V}_{x,n}^{-1}(u) \tag{A.1}$$

[16] See section 6 in [80] for some attempts to fill this gap.

[17] We should mention that the transfer matrices in section 3.1 are not exactly the same we have in this appendix but can be obtained from these via a trivial rescaling in u which obviously does not spoil the invariance of these objects.

where γ is a path starting from (M,K) and finishing at $(0,0)$ (always approaching this point with each step) in a rectangular lattice of size $M\times K$ as in figure 1,[18] $x=(m,k)$ is point in this path and $n=(0,-1)$ or $(-1,0)$ is the unit vector looking along the next step of the path. Each path describes in this way a possible Dynkin diagram of the SU$(K|M)$ super group with corners denoting fermionic nodes and straight lines bosonic ones, see figure 1. Finally,

$$\hat{V}^{-1}_{(m,k),(0,-1)}(u) = e^{i\phi_k}\,\frac{Q_{k,m}(u+i(m-k-1)/2)}{Q_{k,m}(u+i(m-k+1)/2)}\frac{Q_{k-1,m}(u+i(m-k+2)/2)}{Q_{k-1,m}(u+i(m-k+0)/2)} - e^{i\partial_u}$$

$$\hat{V}^{-1}_{(m,k),(-1,0)}(u) = \left(e^{i\varphi_m}\,\frac{Q_{k,m-1}(u+i(m-k-2)/2)}{Q_{k,m-1}(u+i(m-k+0)/2)}\frac{Q_{k,m}(u+i(m-k+1)/2)}{Q_{k,m}(u+i(m-k-1)/2)} - e^{i\partial_u}\right)^{-1}$$

where $Q_{k,m}$ is the Baxter polynomial for the roots of the corresponding node[19] and $\{\phi_k,\varphi_m\}$ are twists introduced in the transfer matrix [40]. Let us then consider a bosonic node like the one in the middle of figure 1 (the *vertical* bosonic node is treated in the same fashion). If the position of this node on the $M\times K$ lattice is given by (m,k) then it is obvious that the only combination containing $Q_{m,k}$ in the right hand side of (A.1) comes from the product of $\hat{V}^{-1}_{(m,k),(-1,0)}(u)\hat{V}^{-1}_{(m+1,k),(-1,0)}(u)$ which reads

$$\Bigg[e^{i\varphi_m+\varphi_{m+1}}\frac{Q_{k,m+1}(u+i(m-k+2)/2)}{Q_{k,m+1}(u+i(m-k+0)/2)}\frac{Q_{k,m-1}(u+i(m-k-2)/2)}{Q_{k,m-1}(u+i(m-k+0)/2)} + e^{2i\partial_u} - \\ - \left(e^{i\varphi_{m+1}}\frac{Q_{k,m}(u+i(m-k-1)/2)}{Q_{k,m}(u+i(m-k+1)/2)}\frac{Q_{k,m+1}(u+i(m-k+2)/2)}{Q_{k,m+1}(u+i(m-k+0)/2)} + \right. \\ \left. + e^{i\varphi_m}\frac{Q_{k,m-1}(u+i(m-k+0)/2)}{Q_{k,m-1}(u+i(m-k+2)/2)}\frac{Q_{k,m}(u+i(m-k+3)/2)}{Q_{k,m}(u+i(m-k+1)/2)}\right)e^{i\partial_u}\Bigg]^{-1} \quad \text{(A.2)}$$

So, if we want to study the bosonic duality on the node (k,m) and its relation with the invariance of several transfer matrices we need to study the last two lines of this expression. For simplicity let us shift u, omit the subscript k in the Baxter polynomials $Q_{k,m-1}, Q_{k,m}, Q_{k,m+1}$ and define the reduced transfer matrix as

$$t(u,\varphi_m,\varphi_{m+1}) \equiv e^{i\varphi_{m+1}}\frac{Q_m(u-i)}{Q_m(u)}\frac{Q_{m+1}(u+i/2)}{Q_{m+1}(u-i/2)} + e^{i\varphi_m}\frac{Q_{m-1}(u-i/2)}{Q_{m-1}(u+i/2)}\frac{Q_m(u+i)}{Q_m(u)}\,. \quad \text{(A.3)}$$

Notice that the absence of poles at the zeros of Q_m yields precisely the Bethe equations for this auxiliary node.

Bosonic duality $\Rightarrow$ Transfer matrices invariance

Thus, to check the invariance of the transfer matrices in all representations it suffices to verify that the reduced transfer matrix $t(u,\varphi_m,\varphi_{m+1})$ is invariant under $\varphi_m \leftrightarrow \varphi_{m+1}$ and

[18] Notice that the path goes in opposite direction compared to the labelling a of the Baxter polynomial Q_a used before. In the notation of this section $Q_{k,m}$ corresponds to the node is at position (m,k) in this lattice.

[19] $\hat{Q}_{0,0}$ is normalized to 1. If we are considering a spin in the representation where the first Dynkin node has a nonzero Dynkin label then $Q_{M,K}$ will play the role of the potential term. In general the situation is more complicated, see [24]. In any case we are mainly interested in the dualization of roots which are not momentum carrying thus we need not care about such matters.

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$Q_m \to \tilde{Q}_m$ where

$$2i\sin\left(\frac{\varphi_{m+1}-\varphi_m}{2}\right) Q_{m-1}(u)Q_{m+1}(u) = \qquad \text{(A.4)}$$
$$e^{i\frac{\varphi_{m+1}-\varphi_m}{2}} Q_m(u-i/2)\tilde{Q}_m(u+i/2) - e^{-i\frac{\varphi_{m+1}-\varphi_m}{2}} Q_m(u+i/2)\tilde{Q}_m(u-i/2)\,.$$

which can be easily verified. If suffices to replace, in $t(u,\varphi_m,\varphi_{m+1})$ in (A.3),

$$\frac{Q_m(u-i)}{Q_m(u)} \to e^{-i(\varphi_{m+1}-\varphi_m)}\frac{\tilde{Q}_m(u-i)}{\tilde{Q}_m(u)} + 2ie^{-i\frac{\varphi_{m+1}-\varphi_m}{2}}\sin\left(\frac{\varphi_{m+1}-\varphi_m}{2}\right)\frac{Q_{m-1}(u+i/2)Q_{m+1}(u+i/2)}{Q_m(u)\tilde{Q}_m(u)}\,,$$
$$\frac{Q_m(u+i)}{Q_m(u)} \to e^{+i(\varphi_{m+1}-\varphi_m)}\frac{\tilde{Q}_m(u+i)}{\tilde{Q}_m(u)} - 2ie^{-i\frac{\varphi_{m+1}-\varphi_m}{2}}\sin\left(\frac{\varphi_{m+1}-\varphi_m}{2}\right)\frac{Q_{m-1}(u-i/2)Q_{m+1}(u-i/2)}{Q_m(u)\tilde{Q}_m(u)}\,,$$

which are obvious consequences of the bosonic duality.

Transfer matrix invariance ⇒ Bosonic duality

On the other hand suppose we have two solutions of Bethe equations, one of them characterized by the Baxter polynomials $\{\ldots,Q_{m-1},Q_m,Q_{m+1},\ldots\}$ with twists $\{\ldots,\varphi_m,\varphi_{m+1},\ldots$ and another with $\{\ldots,Q_{m-1},\tilde{Q}_m,Q_{m+1},\ldots\}$ with twists $\{\ldots,\varphi_{m+1},\varphi_m,\ldots\}$ for which the transfer matrices are the same, that is

$$t(u,\varphi_m,\varphi_{m+1}) = \tilde{t}(u,\varphi_{m+1},\varphi_m)\,. \qquad \text{(A.5)}$$

Then we can show that these two solutions are related by the bosonic duality (A.4). Indeed if we build the Wronskian[20] like object

$$W(u) \equiv e^{i\frac{\varphi_{m+1}-\varphi_m}{2}}\frac{Q_m(u-i/2)\tilde{Q}_m(u+i/2)}{Q_{m-1}(u)Q_{m+1}(u)} - e^{-i\frac{\varphi_{m+1}-\varphi_m}{2}}\frac{Q_m(u+i/2)\tilde{Q}_m(u-i/2)}{Q_{m-1}(u)Q_{m+1}(u)}\,.$$

we can easily check that

$$W(u+i/2)-W(u-i/2) = -e^{-i\frac{\varphi_{m+1}+\varphi_m}{2}}\frac{Q_m(u)\tilde{Q}_m(u)}{Q_{m-1}(u-i/2)Q_{m+1}(u+i/2)}\left(t(u,\varphi_m,\varphi_{m+1})-\tilde{t}(u,\varphi_{m+1},\varphi_m)\right) = 0$$

Since by definition $W(u)$ is a rational function this means it must be a constant. Thus if $\varphi_m \neq \varphi_{m+1}$ we must have $K_m + \tilde{K}_m = K_m + K_{m+1}$ and the value of W can be read from the large u behavior. In this way we obtain precisely the bosonic duality (A.4). If $\varphi_m = \varphi_{m+1}$ then we see that $K_m + \tilde{K}_m = K_m + K_{m+1} + 1$ and we will obtain a different value for the constant W which will correspond to the untwisted bosonic duality described in section 5.3.2.

[20] We would like to thank A.Zabrodin and V.Kazakov for sugesting this nice interpertation for the bosonic duality

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B. Fluctuations for $su(n)$ spin chains

In this appendix we consider a $su(n)$ NBA with the Dynkin labels V_a being $+1$ for a particular a only (the generalization is obvious). This example is obviously more general than that considered in section 4.1 and can be a useful warmup for section 6.5 where we find the integral equation describing the $AdS_5 \times S^5$ 1-loop quantization. For the spin chain $su(n)$ NBA, in the classical limit, we will have n quasi-momenta each one above or below each of the $n-1$ Dynkin nodes.[21] We label these quasi-momenta by p_i (p_j) with i, i' (j, j') taking positive (negative) values for quasi-momenta above (below) the node for which $V_a \neq 0$. Then let us mention how the equations in the previous section are generalized. We consider a *middle node* cut $\mathcal{C}_{1,-1}$. The analogue of equation (4.6) is now

$$-\frac{1}{x} + \sum_j \int\limits_{\mathcal{C}_{1,j}} \frac{\delta\rho(y)}{x-y} + \sum_i \int\limits_{\mathcal{C}_{i,-1}} \frac{\delta\rho(y)}{x-y} + \sum_{n=-N}^{N} \frac{1}{2L} \left[\sum_i \frac{1}{x - x_n^{i,-1}} + \sum_j \frac{1}{x - x_n^{1,j}} \right] = 0 \quad \text{(B.1)}$$

and the charges (4.7), (4.8), (4.9) and (4.11) become[22]

$$Q_r - \int\limits_{\mathcal{C}} \frac{\rho(y)}{y^r} dy = +\sum_n \sum_{ij} \frac{1}{2L} \frac{1}{(x_n^{ij})^r} = +\frac{1}{2L} \sum_{ij} \frac{1}{2J} \oint_{x_n^{ij}} \frac{\cot_{ij}}{y^r} \frac{dy}{2\pi i} \quad \text{(B.2)}$$

$$= +\frac{1}{2L} \sum_{ii'j} \oint_{\mathcal{C}_{i'j}} \frac{\cot_{ij}}{y^r} \frac{dy}{2\pi i} + \frac{1}{2L} \sum_{ijj'} \oint_{\mathcal{C}_{ij'}} \frac{\cot_{ij}}{y^r} \frac{dy}{2\pi i} \quad \text{(B.3)}$$

$$= -\frac{1}{2L} \sum_{ii'j} \oint_{\mathcal{C}_{i'j}} \frac{\cot_{ii'}}{y^r} \frac{dy}{2\pi i} - \frac{1}{2L} \sum_{ijj'} \oint_{\mathcal{C}_{ij'}} \frac{\cot_{jj'}}{y^r} \frac{dy}{2\pi i} \quad \text{(B.4)}$$

$$= -\frac{1}{2L} \int_{\mathcal{C}} \frac{\sum_{i<i'} \Delta \cot_{ii'} + \sum_{j<j'} \Delta \cot_{jj'}}{y^r} \frac{dy}{2\pi i}\,, \quad \text{(B.5)}$$

so that the natural definition of the dressed density becomes now

$$\varrho = \rho + \frac{1}{4L\pi i} \Delta \left(\sum_{i<i'} \cot_{ii'} + \sum_{j<j'} \cot_{jj'} \right) . \quad \text{(B.6)}$$

Next step is to rewrite the integral equation (B.1) in terms of this new density. We proceed exactly as in (4.13), (4.14) using now

$$\cot_{1,i} = -\sum_j \left(\mathcal{I}_{1,j}^{1i} + \mathcal{I}_{i,j}^{1i} \right) \quad , \quad \mathcal{I}_{ij}^{kl} \equiv \int_{\mathcal{C}_{ij}} \frac{\cot_{kl}(y)}{x-y} \frac{dy}{2\pi i}\,,$$

which is the analog of (3.14) for this $su(n)$ setup, so that at the end we obtain the following equation

$$\sum_j \int\limits_{\mathcal{C}_{1,j}} \frac{\delta\varrho(y)}{x-y} + \sum_i \int\limits_{\mathcal{C}_{i,-1}} \frac{\delta\varrho(y)}{x-y} + \frac{1}{L} \left(\cot_{1,-1} - \sum_{ij} \int_{\mathcal{C}_{ij}} \frac{\Delta \cot_{1,i} + \Delta \cot_{j,-1}}{x-y} \frac{dy}{2\pi i} \right) = 0 \quad \text{(B.7)}$$

[21] See figure 11 for an example of such pattern for a super group which clearly resembles $su(8)$.

[22] as in the previous section, we are ignoring the regularization of the charges coming from the contribution of the contour around the origin which would appear in the second line from opening the contours around the excitation points x_n^{ij}.

for $\delta\varrho = \varrho - \varrho_0$ where ϱ_0 obeys the leading order equation

$$-\frac{1}{x} + \sum_j \int_{\mathcal{C}_{1,j}} \frac{\varrho_0(y)}{x-y} + \sum_i \int_{\mathcal{C}_{i,-1}} \frac{\varrho_0(y)}{x-y} = 2\pi k_{1,-1}\,. \tag{B.8}$$

This corrected equation is precisely the one we would obtain from finite size corrections to the $su(n)$ NBA equations. To find this equation from the Bethe ansatz point of view one can simply repeat either of the derivations in section 3, that is the known transfer matrices in various representations or the bosonic duality described in the previous sections. In section 6 we consider the AdS/CFT Bethe ansatz equations which are based on a large rank symmetry group, namely PSU(2, 2|4). There one can see an example of how this could be done in practice (we will only use the dualities approach because at present we don't have the PSU(2, 2|4) transfer matrices for this (exotic) Bethe ansatz equations.).

References

[1] H. Bethe, *On the theory of metals.* 1. *Eigenvalues and eigenfunctions for the linear atomic chain*, *Z. Phys.* **71** (1931) 205.

[2] M. Lüscher, *Volume dependence of the energy spectrum in massive quantum field theories.* 1. *Stable particle states*, *Commun. Math. Phys.* **104** (1986) 177.

[3] T.R. Klassen and E. Melzer, *On the relation between scattering amplitudes and finite size mass corrections in QFT*, *Nucl. Phys.* **B 362** (1991) 329.

[4] B. Sutherland, *Low-lying eigenstates of the one-dimensional Heisenberg ferromagnet for any magnetization and momentum*, *Phys. Rev. Lett.* **74** (1995) 816.

[5] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010 [hep-th/0306139].

[6] O. Babelon, D. Bernard and M. Talon, *Introduction to classical integrable systems*, Cambridge University Press, Cambridge U.K. (2003).

[7] E.D. Belokolos, A.I. Bobenko, V.Z. Enolskii, A.R. Its and V.B. Matveev, *Algebrogeometric approach to nonlinear integrable equations*, Springer, Berlin Germany (1994).

[8] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical / quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[9] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on* $AdS_5 \times S^5$, *Commun. Math. Phys.* **263** (2006) 659 [hep-th/0502226].

[10] N. Gromov, V. Kazakov, K. Sakai and P. Vieira, *Strings as multi-particle states of quantum* σ*-models*, *Nucl. Phys.* **B 764** (2007) 15 [hep-th/0603043].

[11] N. Beisert and A.A. Tseytlin, *On quantum corrections to spinning strings and Bethe equations*, *Phys. Lett.* **B 629** (2005) 102 [hep-th/0509084].

[12] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *Quantum corrections to spinning strings in* $AdS_5 \times S^5$ *and Bethe ansatz: a comparative study*, *JHEP* **09** (2005) 051 [hep-th/0507189].

[13] N. Beisert, A.A. Tseytlin and K. Zarembo, *Matching quantum strings to quantum spins: one-loop vs. finite-size corrections*, *Nucl. Phys.* **B 715** (2005) 190 [hep-th/0502173].

JHEP04(2008)046

[14] R. Hernandez, E. Lopez, A. Perianez and G. Sierra, *Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings*, *JHEP* **06** (2005) 011 [hep-th/0502188].

[15] N. Beisert and L. Freyhult, *Fluctuations and energy shifts in the Bethe ansatz*, *Phys. Lett.* **B 622** (2005) 343 [hep-th/0506243].

[16] N. Gromov and V. Kazakov, *Double scaling and finite size corrections in* SL(2) *spin chain*, *Nucl. Phys.* **B 736** (2006) 199 [hep-th/0510194].

[17] P.Y. Casteill and C. Kristjansen, *The strong coupling limit of the scaling function from the quantum string Bethe ansatz*, *Nucl. Phys.* **B 785** (2007) 1 [arXiv:0705.0890].

[18] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *Complete spectrum of long operators in* $N = 4$ *SYM at one loop*, *JHEP* **07** (2005) 030 [hep-th/0503200].

[19] P.A. Bares, I.M.P. Karmelo, J. Ferrer and P. Horsch, *Charge-spin recombination in the one-dimensional supersymmetric* $t - J$ *model*, *Phys. Rev.* **46** (1992) 14624.

[20] F.H.L. Essler, V.E. Korepin and K. Schoutens, *New exactly solvable model of strongly correlated electrons motivated by high* T_c *superconductivity*, cond-mat/9209002; *Exact solution of an electronic model of superconductivity in* $(1 + 1)$*-dimensions.* 1, cond-mat/9211001.

[21] F. Göhmann and A. Seel, *A note on the Bethe ansatz solution of the supersymmetric* $t - J$ *model*, cond-mat/0309138.

[22] F. Woynarovich, *Low energy excited states in a Hubbard chain with on-site attraction*, *J. Phys.* **C 16** (1983) 6593.

[23] Z. Tsuboi, *Analytic Bethe ansatz and functional equations associated with any simple root systems of the Lie superalgebra* $\mathrm{SL}(r + 1|s + 1)$, *Physica* **A 252** (1998) 565.

[24] V. Kazakov, A. Sorin and A. Zabrodin, *Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics*, *Nucl. Phys.* **B 790** (2008) 345 [hep-th/0703147].

[25] N. Beisert and M. Staudacher, *Long-range* $\mathrm{PSU}(2, 2|4)$ *Bethe ansaetze for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [hep-th/0504190].

[26] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[27] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[28] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[29] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for* $N = 4$ *super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[30] N. Beisert, *The complete one-loop dilatation operator of* $N = 4$ *super Yang-Mills theory*, *Nucl. Phys.* **B 676** (2004) 3 [hep-th/0307015].

[31] N. Gromov and P. Vieira, *The* $AdS_5 \times S^5$ *superstring quantum spectrum from the algebraic curve*, *Nucl. Phys.* **B 789** (2008) 175 [hep-th/0703191].

[32] N. Gromov and P. Vieira, *Constructing the AdS/CFT dressing factor*, *Nucl. Phys.* **B 790** (2008) 72 [hep-th/0703266].

[33] R.A. Janik and T. Lukowski, *Wrapping interactions at strong coupling — the giant magnon*, *Phys. Rev.* **D 76** (2007) 126008 [arXiv:0708.2208].

[34] S. Schäfer-Nameki, *Exact expressions for quantum corrections to spinning strings*, *Phys. Lett.* **B 639** (2006) 571 [hep-th/0602214].

[35] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *How accurate is the quantum string Bethe ansatz?*, *JHEP* **12** (2006) 020 [hep-th/0610250].

[36] G. Arutyunov, S. Frolov and M. Zamaklar, *Finite-size effects from giant magnons*, *Nucl. Phys.* **B 778** (2007) 1 [hep-th/0606126].

[37] J. Ambjørn, R.A. Janik and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, *Nucl. Phys.* **B 736** (2006) 288 [hep-th/0510171].

[38] S. Schäfer-Nameki and M. Zamaklar, *Stringy sums and corrections to the quantum string Bethe ansatz*, *JHEP* **10** (2005) 044 [hep-th/0509096].

[39] R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, *JHEP* **07** (2006) 004 [hep-th/0603204].

[40] A. Zabrodin, *Backlund transformations for difference Hirota equation and supersymmetric Bethe ansatz*, arXiv:0705.4006.

[41] L.D. Faddeev, *How algebraic Bethe ansatz works for integrable model*, hep-th/9605187.

[42] N.Y. Reshetikhin, *A method of functional equations in the theory of exactly solvable quantum systems*, *Lett. Math. Phys.* **7** (1983) 205.

[43] N.Y. Reshetikhin, *Integrable models of quantum one-dimensional magnets with $O(N)$ and* $\mathrm{Sp}(2K)$ *symmetry*, *Theor. Math. Phys.* **63** (1985) 555 [*Teor. Mat. Fiz.* **63** (1985) 347].

[44] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, *Quantum integrable models and discrete classical Hirota equations*, *Commun. Math. Phys.* **188** (1997) 267 [hep-th/9604080].

[45] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in $AdS_5 \times S^5$*, *Nucl. Phys.* **B 668** (2003) 77 [hep-th/0304255].

[46] S. Frolov and A.A. Tseytlin, *Quantizing three-spin string solution in $AdS_5 \times S^5$*, *JHEP* **07** (2003) 016 [hep-th/0306130].

[47] G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: new integrable system relations*, *Phys. Rev.* **D 69** (2004) 086009 [hep-th/0311004].

[48] I.Y. Park, A. Tirziu and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: one-loop correction to energy in* $\mathrm{SL}(2)$ *sector*, *JHEP* **03** (2005) 013 [hep-th/0501203].

[49] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[50] S. Randjbar-Daemi, A. Salam and J.A. Strathdee, *Generalized spin systems and σ-models*, *Phys. Rev.* **B 48** (1993) 3190 [hep-th/9210145].

[51] M. Kruczenski, *Spin chains and string theory*, *Phys. Rev. Lett.* **93** (2004) 161602 [hep-th/0311203].

[52] M. Kruczenski, A.V. Ryzhov and A.A. Tseytlin, *Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains*, *Nucl. Phys.* **B 692** (2004) 3 [hep-th/0403120].

JHEP04(2008)046

[53] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *JHEP* **06** (2002) 007 [hep-th/0204226].

[54] T. Bargheer, N. Beisert and N. Gromov, to appear.

[55] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[56] M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **05** (2005) 054 [hep-th/0412188].

[57] N. Beisert, *The* SU(2|2) *dynamic S-matrix*, hep-th/0511082.

[58] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [hep-th/0609044].

[59] N. Beisert, *On the scattering phase for $AdS_5 \times S^5$ strings*, *Mod. Phys. Lett.* **A 22** (2007) 415 [hep-th/0606214].

[60] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry*, *Phys. Rev.* **D 73** (2006) 086006 [hep-th/0603038].

[61] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [hep-th/0610251].

[62] B. Eden and M. Staudacher, *Integrability and transcendentality*, *J. Stat. Mech.* (2006) P11014 [hep-th/0603157].

[63] A.V. Kotikov and L.N. Lipatov, *DGLAP and BFKL equations in the $N = 4$ supersymmetric gauge theory*, *Nucl. Phys.* **B 661** (2003) 19 [*Erratum ibid.* **B 685** (2004) 405] [hep-ph/0208220].

[64] Z. Bern, M. Czakon, L.J. Dixon, D.A. Kosower and V.A. Smirnov, *The four-loop planar amplitude and cusp anomalous dimension in maximally supersymmetric Yang-Mills theory*, *Phys. Rev.* **D 75** (2007) 085010 [hep-th/0610248].

[65] N. Beisert, T. McLoughlin and R. Roiban, *The four-loop dressing phase of $N = 4$ SYM*, *Phys. Rev.* **D 76** (2007) 046002 [arXiv:0705.0321].

[66] A.V. Kotikov, L.N. Lipatov, A. Rej, M. Staudacher and V.N. Velizhanin, *Dressing and wrapping*, *J. Stat. Mech.* (2007) P1010003 [arXiv:0704.3586].

[67] N. Dorey, D.M. Hofman and J.M. Maldacena, *On the singularities of the magnon S-matrix*, *Phys. Rev.* **D 76** (2007) 025011 [hep-th/0703104].

[68] M. Kruczenski, R. Roiban, A. Tirziu and A.A. Tseytlin, *Strong-coupling expansion of cusp anomaly and gluon amplitudes from quantum open strings in $AdS_5 \times S^5$*, *Nucl. Phys.* **B 791** (2008) 93 [arXiv:0707.4254].

[69] R. Roiban and A.A. Tseytlin, *Strong-coupling expansion of cusp anomaly from quantum superstring*, *JHEP* **11** (2007) 016 [arXiv:0709.0681].

[70] T. Klose and K. Zarembo, *Reduced σ-model on $AdS_5 \times S^5$: one-loop scattering amplitudes*, *JHEP* **02** (2007) 071 [hep-th/0701240].

[71] T. Klose, T. McLoughlin, J.A. Minahan and K. Zarembo, *World-sheet scattering in $AdS_5 \times S^5$ at two loops*, *JHEP* **08** (2007) 051 [arXiv:0704.3891].

[72] J.M. Maldacena and I. Swanson, *Connecting giant magnons to the pp-wave: an interpolating limit of $AdS_5 \times S^5$*, *Phys. Rev.* **D 76** (2007) 026002 [hep-th/0612079].

[73] L. Freyhult, *Bethe ansatz and fluctuations in* SU(3) *Yang-Mills operators*, *JHEP* **06** (2004) 010 [hep-th/0405167].

[74] L. Freyhult and C. Kristjansen, *Finite size corrections to three-spin string duals*, *JHEP* **05** (2005) 043 [hep-th/0502122].

[75] L. Freyhult and C. Kristjansen, *A universality test of the quantum string Bethe ansatz*, *Phys. Lett.* **B 638** (2006) 258 [hep-th/0604069].

[76] N. Beisert and R. Roiban, *Beauty and the twist: the Bethe ansatz for twisted $N = 4$ SYM*, *JHEP* **08** (2005) 039 [hep-th/0505187].

[77] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for (non)supersymmetric deformations of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 731** (2005) 1 [hep-th/0507021].

[78] H.-Y. Chen, N. Dorey and R.F. Lima Matos, *Quantum scattering of giant magnons*, *JHEP* **09** (2007) 106 [arXiv:0707.0668].

[79] D.M. Hofman and J.M. Maldacena, *Giant magnons*, *J. Phys.* **A 39** (2006) 13095 [hep-th/0604135].

[80] N. Beisert, *The analytic Bethe ansatz for a chain with centrally extended* SU(2|2) *symmetry*, *J. Stat. Mech.* (2007) P01017 [nlin/0610017].